Equivalences between Calabi–Yau manifolds and roofs of projective bundles

by

Marco Rampazzo

Thesis submitted in fulfilment of
the requirements for the degree of
PHILOSOPHIAE DOCTOR
(PhD)

University of Stavanger

Faculty of Science and Technology
Department of Mathematics and Physics
2021

University of Stavanger
NO-4036 Stavanger
NORWAY
www.uis.no

ISBN: 978-82-8439-001-7
ISSN: 1890-1387
PhD: Thesis UiS No. 585

Preface

This thesis is submitted in partial fulfilment of the requirements for the degree of Philosophiae Doctor (PhD) at the University of Stavanger, Faculty of Science and Technology, Norway. The research has been carried out at the University of Stavanger from September 2016 to October 2020.

I would like to express the most sincere gratitude to my advisor Michał Kapustka for his infinite patience and support, for all the time we spent working together and for everything he taught me so far, this work would not have been possible without his guide. I am also deeply grateful to Laurent Manivel for proposing me new problems and valuable insight during my stay at the University of Toulouse, and to Sasha Kuznetsov for providing very useful advice and ideas while reading early drafts of some of my work.
I owe some words of gratitude towards my fellow PhD students and colleagues here at the University of Stavanger for the friendly atmosphere they contributed to create, and to the institution itself for the excellent working environment.
I wish to thank my family for being always present, regardless of the physical distance. Lastly, I want to express my deepest thanks to my wife Francesca.

Marco Rampazzo

Stavanger, February 2021

Abstract

It is conjectured that many birational transformations, called K-inequalities, have a categorical counterpart in terms of an embedding of derived categories. In the special case of simple K-equivalence (or more generally K-equivalence), a derived equivalence is expected: we propose a method to prove derived equivalence for a wide class of such cases. This method is related to the construction of roofs of projective bundles introduced by Kanemitsu. Such roofs can be related to candidate pairs of derived equivalent, $\mathbb{L}$-equivalent and non isomorphic Calabi–Yau varieties, we prove such claims in some examples of this construction.

In the same framework, we show that a similar approach applies to prove derived equivalence of pairs of Calabi–Yau fibrations, we provide some working examples and we relate them to gauged linear sigma model phase transitions.

Table of Contents

1 Introduction

Calabi—Yau varieties have been object of intense research through the last decades. Due to their spontaneous appearance in the context of a geometric approach to explain fundamental physics, and their special place in the classification of complex varieties, they gathered the attention of mathematicians and physicists from diverse fields: a famous example is the pioneering work (CDGP91) which led to the development of mirror symmetry, and eventually to Kontsevich's homological mirror symmetry conjecture.

On the other hand, an interesting open problem is to determine how far we can interpret the derived category of coherent sheaves as an invariant: for instance, in light of the reconstruction theorem of Bondal and Orlov (BO01), Fano and general type varieties are isomorphic if and only if their derived category is equivalent. However, while still quite uncommon, there exist several examples of pairs of Calabi—Yau varieties which are derived equivalent but not isomorphic, or even not birationally equivalent. In fact, in dimension three, birational equivalence determines derived equivalence (Bri02), but the role of derived category as a birational invariant for higher dimensional Calabi–Yau varieties has not been clarified yet.

Even beyond the world of Calabi–Yau varieties, one of the most promising ideas in this field is the so-called DK-conjecture by Bondal, Orlov and Kawamata (BO02; Kaw02): the conjecture says that two smooth varieties X_1 and X_2 are expected to be derived equivalent if there exists a birational morphism $\mu : X_1 \dashrightarrow X_2$ resolved by two morphisms $f_1 : X \longrightarrow X_1$ and $f_2 : X \longrightarrow X_2$ such that $f_1^*\omega_{X_1} = f_2^*\omega_{X_2}$. This statement is supported by many examples (e.g. (BO02; Nam03; Bri02)).

Nowadays, many pairs of non-trivially derived equivalent pairs of Calabi–Yau varieties have been shown to fit in the homological projective duality or categorical join programs (Kuz07; KP19), like (OR17; BCP20; Man17; Kuz06b; BC08). However, there exists a class of Calabi—Yau pairs for which the proofs of derived equivalence still rely on ad-hoc arguments (IMOU19; Kuz18; KR17; KR20; Muk98), despite their geometry shares many similarities. We propose a general construction to describe them, which leads to a method to prove derived equivalence.

In a recent paper by Kanemitsu (Kan18), in the context of the DK-conjecture, a partial classification has been given for a special class of Fano varieties with two different projective bundle structures, called *roofs*. We show that the data of a general hyperplane section on such varieties defines a pair of Calabi—Yau varieties and we conjecture that such pairs are derived equivalent. We motivate the conjecture with many examples, some already present in the literature alongside with several new ones.

A relative version of the problem discussed above yields a pair of Calabi—Yau fibrations: we discuss their derived equivalence in relation to the derived equivalence of the fibers. In particular, we prove that under some assumptions, if a general pair of Calabi—Yau varieties associated to a roof is derived equivalent, the related fibrations over a smooth projective base are derived equivalent as well. This extends a result by Bridgeland and Maciocia, where given a $K3$ fibration of dimension three, a derived equivalent fibration is constructed by replacing each fiber with a two-dimensional moduli space of stable sheaves on the original fiber (BM02). As an example, we construct a pair of Calabi—Yau eightfolds fibered in Calabi—Yau threefolds such that for a general point in the basis the fibers are not birationally equivalent. To this purpose, we introduce a class of locally trivial fibrations of roofs, which we call *homogeneous roof bundles*.

Alongside with their relation with Calabi—Yau fibrations, being a particular class of the families of roofs studied by Kanemitsu in (Kan18), homogeneous roof bundles have an application in the context of the DK-conjecture. A simple K-equivalence map is a birational morphism resolved by two smooth blowups with isomorphic exceptional loci. Kanemitsu proved that in every simple K-equivalence the exceptional locus is isomorphic to a family of roofs over a smooth projective variety. At the price of the additional hypothesis of local triviality, restricting our attention to homogeneous roof bundles allows to approach the problem of derived equivalence with a method based on mutations of exceptional collections. In fact, we show how the data of a homogeneous roof bundle describes three different problems: a pair of Calabi–Yau varieties, a pair of fibrations with Calabi–Yau general fibers and a simple K-equivalent map. In all three cases, we obtain semiorthogonal decompositions which are formally identical and suggest that proving derived equivalence for the simplest setting (the Calabi–Yau pair) allows to extend the result to the latter. In fact, we conjecture that all these three problems give rise to a derived equivalence, and we prove that this is the case under the validity of some additional assumptions.

An interesting example arises from the context of the self-projective dual orbit W of the action of $GL(V_6)$ on $\mathbb{P}(\wedge^3 V_6)$, where V_6 is a vector space of dimension six. W is a fourteen-dimensional Fano variety of index ten, with a nine dimensional singular locus $W_{sing} \simeq G(3, V_6)$. This variety is naturally embedded in $\mathbb{P}(\wedge^3 V_6) \simeq \mathbb{P}^{19}$ and the intersection of two general translates of W is expected to be a (singular) Calabi–Yau variety. In the somewhat similar case of Calabi–Yau intersections of general translates of $G(2, 5)$ in $\mathbb{P}^9$, a degeneration of the family is given by zero loci of sections of the normal bundle of $G(2, 5)$. We construct a similar picture from a desingularization W_0 of W: we obtain pairs of nine-dimensional Calabi–Yau sextuple covers of the flag variety $F(1, 5, V_6)$, which are proved to be derived equivalent by an application of the methods above.
Mirror symmetry conjectures have now been proven on a reasonable level of generality in the case of Calabi–Yau manifolds which are complete intersections in toric varieties: a central role in this setting was played by the abelian Gauged Linear Sigma Models. More precisely, Calabi–Yau manifolds which are complete intersections in toric varieties admit good models as GIT quotients of critical loci of invariant

functions called superpotentials, defined on vector spaces equipped with an action of an abelian gauge group. In particular, changing the stability condition in the GIT quotient leads to new phases of the model. In the case of abelian gauged linear sigma models, the new phases have radically different nature. Still the relation between these phases and the original Calabi–Yau phase has been an important asset to the theory.

Taking into consideration non abelian gauged linear sigma models, at the price of increased complexity, one obtains new interesting phenomena. One of the most intriguing features is the possibility of having multiple Calabi–Yau phases described by the same GLSM: this leads to the existence of Calabi–Yau manifolds which are strictly related to each other, but sometimes still not isomorphic or birational. The physical argument of these phases having the same D-brane category translates, in a more mathematical parlance, to a conjectural derived equivalence: Such is the case, for example, of the so-called Pfaffian–Grassmannian pair discovered by Rødland (Rød98), where derived equivalence has been proved later by Borisov and Căldăraru (BC08) and Kuznetsov (Kuz06b). A new proof, inspired by physics, has been given by Addington, Donovan and Segal (ADS15).

It is a natural question to ask whether such GLSM duality can be realized for Calabi–Yau pairs arising from a roof construction. We give a partial answer to the question by establishing a GLSM with two geometric phases isomorphic to a pair of Calabi–Yau varieties (Y_1, Y_2) of dimension k^2-1, where $Y_1 \in G(k, 2k+1)$ and $Y_2 \in G(k+1, 2k+1)$,

are the Calabi–Yau pair associated to the roof of type A^G_{2k} of the list of Kanemitsu (Kan18, Section 5.2.1). For $k = 2$, the resulting pair can be interpreted as a degeneration of the family of intersections of $G(2,5)$ translates in $\mathbb{P}^9$ studied by (OR17; BCP20). This gauged linear sigma model has a particularly simple description in terms of variation of GIT, since both the Calabi–Yau varieties arise as GIT quotients of the critical locus in two different stability chambers. To the best of the author's knowledge, this is the only GLSM with such characteristics. Furthermore, we generalize this model to a GLSM yielding two geometric phases isomorphic to Calabi–Yau varieties of dimension $k^2 + 2k - 1$ which have a description as fibrations of Calabi–Yau varieties on $\mathbb{P}^{2k}$, such that for a general point in $\mathbb{P}^{2k}$ the fibers are isomorphic to the pair (Y_1, Y_2) introduced above.
Finally, we observe that the geometry of roofs has an interesting relation with the D-brane categories of associated Landau–Ginzburg models. In fact, by an application of Knörrer periodicity (Shi12), a derived equivalence of a Calabi–Yau pair described by a roof construction lifts to an equivalence of matrix factorization category of total spaces of vector bundles with appropriate superpotentials.

Notations and conventions

- We work over the field of complex numbers.
- We shall use the notation $\mathbb{P}(\mathcal{E}) := \mathrm{Proj}(\mathrm{Sym}\,\mathcal{E})$, whenever $\mathcal{E}$ is a vector bundle or a vector space.
- The orbit of an element v with respect to a group G is denoted by

$G.v$, while the image of v under the action of $g \in G$ is denoted by $g.v$.

- Given a vector space V, the contraction of $v \in V$ with $w \in V^\vee$ is denoted by $v \cdot w$.
- With the expression $Z(\sigma)$ we mean the zero locus of a section σ of a vector bundle.
- Given a vector space V and $k \in \mathbb{Z}$, we call $V[k]$ the complex of vector spaces which is identically zero in every degree except for $-k$, where it is equal to V. For example, $H^\bullet(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = \mathbb{C}[0]$.
- The expression $A \times^G B$ denotes the balanced product (see Section 2.3), to be distinguished by the fiber product $A \times_G B$.
- In the context of triangulated categories, we will often refer to a triangle as $A \longrightarrow B \longrightarrow C$, with the meaning of $A \longrightarrow B \longrightarrow C \longrightarrow A[1] \longrightarrow B[1] \longrightarrow C[1] \longrightarrow \cdots$

2 Homogeneous varieties and vector bundles

2.1 Algebraic groups

In this section we will summarize some basic definitions about algebraic groups.

Definition 2.1.1. *(Bor91, 1.1) An* algebraic group *is an algebraic variety G together with*

- *an identity element $e \in G$*
- *a morphism*

$$\begin{aligned} \mu : G \times G &\longrightarrow G \\ (x, y) &\longmapsto xy \end{aligned} \tag{2.1.1}$$

- *a morphism*

$$\begin{aligned} i : G &\longrightarrow G \\ x &\longmapsto x^{-1} \end{aligned} \tag{2.1.2}$$

with respect to which the set G is a group.

Morphisms of algebraic groups are morphisms of algebraic varieties which commute with the group operations, while algebraic subgroups are Zariski-closed subvarieties which are also closed under the group operation.
From now on, to simplify the notation, let us drop the operation μ and denote an algebraic group just by its underlying algebraic variety.

Definition 2.1.2. *(Bor91, Section 1.6) Let V be a vector space. An algebraic group G is called* linear *if it is a closed subgroup of $GL(V)$.*

Definition 2.1.3. *(CGP15, Definition A.1.15) A linear algebraic group G is called* reductive *if every smooth connected unipotent normal subgroup is trivial.*

Remark 2.1.4. There exists a scheme-theoretic version of the definition of algebraic groups (see e.g. (Mil17)). However, since we will mostly deal with smooth algebraic varieties, we prefer to keep a more elementary formalism where possible.

The following definitions are standard (see for example (Ott95)):

Definition 2.1.5. *An* action *of algebraic group G on an algebraic variety X is an algebraic morphism*

$$\begin{aligned} \alpha : G \times X &\longrightarrow X \\ (g, x) &\longmapsto gx \end{aligned} \tag{2.1.3}$$

such that for every $x \in X$ one has $Ix = x$, and for every $g_1, g_2 \in G$ and $x \in X$ one has $(g_1 g_2)x = g_1(g_2 x)$.

Definition 2.1.6. *An algebraic variety is called homogeneous if it admits a transitive action by an algebraic group.*

2.2 Parabolic subgroups and homogeneous varieties

In this section we will give a quick review on rational homogeneous varieties and their description in terms of quotients of algebraic groups.

Definition 2.2.1. *(Bor91, 11.2) Let G be a connected affine group. A* parabolic subgroup *$P \subset G$ is a closed subgroup such that the quotient G/P is a complete variety.*

One observes that a homogeneous space of the form G/P is always quasi-projective (Bor91, 6.8), then G/P is a projective variety if and only if P is parabolic. Furthermore, for G linear, reduced and connected, G/P is Fano for every P (Kol96, Chapter V, Theorem 1.4).

Algebraic groups of the form G/P, where G is simple and P is parabolic, are particularly interesting due to the following result, due to Borel and Remmert:

Theorem 2.2.2. *A rational homogeneous variety X is isomorphic to a product*

$$X = G_1/P_1 \times \cdots \times G_n/P_n \tag{2.2.1}$$

where G_i is a simple group and P_i is a parabolic subgroup for $1 \leq i \leq n$.

2.2.1 Basic Lie algebra theory

Let us first recall some basic information on Lie algebras, which can be found, for example, in (Cor97), and fix the notation for the next paragraphs.

Cartan subalgebras

Definition 2.2.3. *(Hel62, Ch. III, Sec. 3) Let $\mathfrak{g}$ be a complex semisimple Lie algebra. A* Cartan subalgebra *$\mathfrak{h} \subset \mathfrak{g}$ is a subalgebra satisfying the following conditions:*

- *it is a maximal abelian subalgebra of* $\mathfrak{g}$
- *the adjoint representation* $\boldsymbol{ad}(h)$ *is completely reducible for every element* $h \in \mathfrak{h}$.

Such subalgebra always exists (Hel62, Theorem 4.1) and it is unique up to automorphism of $\mathfrak{g}$ (Hel62, Ch. III, Sec. 5)

Let us consider a basis $\{h_1, \cdots, h_r\}$ of the Cartan subalgebra $\mathfrak{h}$ of an n-dimensional semisimple Lie algebra $\mathfrak{g}$ of rank r. Since $\mathfrak{h}$ is abelian, then the matrices $\{\mathbf{ad}(h_1), \ldots, \mathbf{ad}(h_r)\}$ are simultaneously diagonalizable. Hence, there exists a basis $\{h_1, \cdots, h_r, a_{r+1}, \cdots a_n\}$ of $\mathfrak{g}$ such that the action of $\mathbf{ad}(h_i)$ acts on the a_j's as multiplication by constant. Thus, since by definition $\mathbf{ad}(h_i)(-) = [h_i, -]$, we can introduce the notation

$$[h_i, a_j] = \alpha_j(h_i)a_j. \tag{2.2.2}$$

By linearity, Equation 2.2.2 defines for every basis element a_i a linear functional called a *root* of $\mathfrak{g}$ with respect to $\mathfrak{h}$ (Cor97, Chapter 13, Section 4):

$$\begin{aligned} \alpha_i : \mathfrak{h} &\longrightarrow \mathbb{C} \\ h &\longmapsto \alpha_i(h). \end{aligned} \tag{2.2.3}$$

Let us now consider the *Killing form*

$$K(x, y) := \mathrm{tr}(\mathbf{ad}(x)\mathbf{ad}(y)). \tag{2.2.4}$$

Once restricted to $\mathfrak{h}$ it provides a nondegenerate symmetric bilinear form (Cor97, Chapter 13, Theorem III), which can be used to associate to each root α a unique element $h_\alpha \in \mathfrak{h}$ by:

$$K(h_\alpha, h) = \alpha(h). \tag{2.2.5}$$

We also introduce the notation $\langle\alpha_1, \alpha_2\rangle := K(h_{\alpha_1}, h_{\alpha_2})$.

Theorem 2.2.4. *(Cor97, Chapter 13, Section 5, Theorem V)*
Let $\{h_{\alpha_1}, \ldots, h_{\alpha_r}\} \subset \mathfrak{h}$ be a linearly independent set, for some roots $\{\alpha_1, \ldots, \alpha_r\}$. Then every root α can be written as $\alpha = k_1\alpha_1 + \cdots + k_r\alpha_r$ with real and rational coefficients $k_1, \ldots, k_r$.

Definition 2.2.5. *(Cor97, Chapter 13, Section 7) Let $\{h_{\alpha_1}, \ldots, h_{\alpha_r}\} \subset \mathfrak{h}$ be a linearly independent set, for some roots $\{\alpha_1, \ldots, \alpha_r\}$. A non zero root α is said to be* positive *with respect to $\{\alpha_1, \ldots, \alpha_r\}$ if the first nonvanishing coefficient k_i of the expression $\alpha = k_1\alpha_1 + \cdots + k_r\alpha_r$ is positive.*

Definition 2.2.6. *(Cor97, Chapter 13, Section 7) A non zero root α is said to be* simple *with respect to $\{\alpha_1, \ldots, \alpha_r\}$ if it cannot be expressed as $\alpha = \alpha' + \alpha''$, where α' and α'' are positive roots with respect to $\{\alpha_1, \ldots, \alpha_r\}$.*

Theorem 2.2.7. *(Cor97, Chapter 13, Section 7, Theorem II)*
Let $\{\alpha_1, \ldots, \alpha_r\}$ be simple roots. Then every positive root α can be written as $\alpha = k_1\alpha_1 + \cdots + k_r\alpha_r$ for nonnegative integers $k_1, \ldots, k_r$.

Hereafter we will denote Δ the set of roots, and $\Delta_\pm$ the subset of positive (negative) roots, while the set of simple roots will be called $\mathcal{S}$.

The classification of semisimple Lie algebras

Let us fix a Lie algebra $\mathfrak{g}$ of rank r with a Cartan subalgebra $\mathfrak{h}$ and a set of simple roots $\mathcal{S} = \{\alpha_1, \cdots, \alpha_r\}$. Then, we can define a $r \times r$

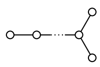
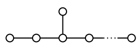
invertible matrix called *Cartan matrix* in the following way:

$$A_{ij} := \frac{2\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_i, \alpha_i \rangle} \tag{2.2.6}$$

Every root of $\mathfrak{g}$ can be recovered by the Cartan matrix and a choice of $\mathcal{S}$ by *Weyl reflections*:

$$S_{\alpha_i} : \alpha_j \longmapsto \alpha_j - A_{ij}\alpha_i \tag{2.2.7}$$

Note that all the information required to characterize a semisimple Lie algebra is encoded in its Cartan matrix. In fact, one can prove that the entries of such matrix can only be integers smaller or equal than three, and that there exist only a finite set of admissible Cartan matrices. This fact leads to the famous Dynkin-type classification of semisimple Lie algebras:

A_n		$n \geq 1$	
B_n		$n \geq 2$	
C_n		$n \geq 2$	
D_n		$n \geq 4$	(2.2.8)
E_n		$6 \leq n \leq 8$	
F_4			
G_2			

Weights and representations

Hereafter we will review how representations of semisimple Lie algebras can be characterized by their highest weight. This will fix background and notation in order to introduce homogeneous vector bundles.

Let us fix a complex semisimple Lie algebra $\mathfrak{g}$ of dimension n and rank r, and a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Furthermore, let us choose a set of simple roots $\mathcal{S} = \{\alpha_1, \cdots, \alpha_r\}$. Let Γ be a N-dimensional representation of $\mathfrak{g}$, i.e. a homomorphism of the following kind:

$$\Gamma : \mathfrak{g} \longrightarrow \operatorname{End} V_\Gamma \tag{2.2.9}$$

where $V_\Gamma \simeq \mathbb{C}^N$. The matrices $\Gamma(h)$ for every $h \in \mathfrak{h}$ can be diagonalized simultaneously (Cor97, Chapter 15, Section 2) hence let us assume they are diagonal. Then, the *weights* of Γ are the following linear functionals:

$$\begin{aligned} \Lambda_j : \mathfrak{h} &\longrightarrow \mathbb{C} \\ h &\longmapsto \Gamma_{jj}(h) \end{aligned} \tag{2.2.10}$$

In other words, evaluations of weights on an element h are eigenvalues of the diagonal operators $\Gamma(h) \in \operatorname{End} V_\Gamma$ (Cor97, Ch. 15, Sec. 2).

Theorem 2.2.8. *(Cor97, Chapter 15, Theorem III) Every weight ω can be written as $\omega = k_1\alpha_1 + \cdots + k_r\alpha_r$, where $\alpha_1, \ldots, \alpha_r$ are simple roots and all coefficients $k_1, \ldots, k_r$ are real and rational.*

Weyl reflections can be extended to weights: in other words, one can define a Weyl reflection $S_\alpha(\omega)$ for a weight ω and a simple root α by expressing ω as linear combinations of simple roots. Moreover, one calls *positive* a weight $\omega = k_1\alpha_1 + \cdots + k_r\alpha_r$ such that the first nonvanishing k_i is positive, and introduce a partial ordering among weights: namely, we say $\omega > \omega'$ if $\omega - \omega'$ is a positive weight. We call *highest weight* of a representation a unique weight Λ such that $\Lambda > \omega$ for every other weight ω.

One defines a distinguished set of *fundamental weights* in the following way (Cor97, Chapter 15, Section 3):

$$\omega_j = \sum_{k=1}^{r} (A^{-1})_{kj} \alpha_k(h) \tag{2.2.11}$$

The highest weight of an irreducible representation can be written as a linear combination of the fundamental weights with nonnegative integeral coefficients (Cor97, Chapter 15, Section 3, Theorem I). In the following, given a representation Γ_ω of highest weight $\omega = \sum_i \lambda_i \omega_i$, we will denote such weight as

$$\omega = (\lambda_1, \ldots, \lambda_r).$$

Definition 2.2.9. *A weight $\lambda = \sum_i \lambda_i \omega_i$ is called* dominant *if $\lambda_i \geq 0$ for every i*

Definition 2.2.10. *We call* length *of a Weyl reflection S the minimal integer $l(S)$ such that S is a composition of $l(S)$ Weyl reflections with respect to s imple roots.*

2.2.2 Levi decomposition and parabolic subgroups

There exists a nice combinatoric description of parabolic subgroups of a linear reductive algebraic group G, which will be reviewed here. We follow (IMOU16) and the sources therein.

Let us fix a linear reductive algebraic group G, and the corresponding Lie algebra $\mathfrak{g}$. Then $\mathfrak{g}$ decomposes in the following direct sum:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha \tag{2.2.12}$$

where the root spaces $\mathfrak{g}_\alpha$ are given by

$$\mathfrak{g}_\alpha = \{g \in \mathfrak{g} : [h, g] = \alpha(h) g \text{ for } h \in \mathfrak{h}\} \tag{2.2.13}$$

Inside $\mathfrak{g}$, one distinguishes the *standard Borel subalgebra* given by the expression

$$\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n} \tag{2.2.14}$$

where $\mathfrak{n} = \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_\alpha$. A useful characterization of parabolic subalgebras is given in terms of subalgebras containing $\mathfrak{b}$. The direct sum decomposition of Equation 2.2.12 allows us to wrote such subalgebras in a particularly convenient way. Given the chosen set $\mathcal{S} = \{\alpha_1, \ldots, \alpha_r\}$ of simple roots, we define subsets $\mathcal{S}_{i_1 \ldots i_l} := \mathcal{S} \setminus \{\alpha_{i_1}; \ldots; \alpha_{i_l}\}$ and $\Delta_{i_1 \ldots i_l} = \mathrm{span}\mathcal{S}_{i_1 \ldots i_l} \cap \Delta$. We then introduce the *levi subalgebras*

$$\mathfrak{l}_{i_1 \ldots i_l} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta_{i_1 \ldots i_l}} \mathfrak{g}_\alpha \tag{2.2.15}$$

One can check that $\mathfrak{l} = [\mathfrak{l}, \mathfrak{l}] + \mathfrak{l}_Z$ where $\mathfrak{l}_Z$ is the center. Moreover $[\mathfrak{l}, \mathfrak{l}]$ is semisimple and its rank is the cardinality of $\mathcal{S}_{i_1 \ldots i_l}$. In a similar way, one can define nilpotent subalgebras of the form

$$\mathfrak{u}_{i_1 \ldots i_l} = \bigoplus_{\alpha \in \Delta^+ \setminus \Delta_{i_1 \ldots i_l}} \mathfrak{g}_\alpha$$

Then the direct sum

$$\mathfrak{p}_{i_1 \ldots i_l} = \mathfrak{l}_{i_1 \ldots i_l} \oplus \mathfrak{u}_{i_1 \ldots i_l}. \tag{2.2.16}$$

is a subalgebra and contains $\mathfrak{b}$, hence it is parabolic. The list of parabolic subalgebras $\{\mathfrak{p}_{i_1,\dots,i_l}\}$ is exhaustive up to conjugation.

Remark 2.2.11. Given a semisimple Lie group G and parabolic subgroup P such that $B \subset P$, the maps

$$\pi : G/B \longrightarrow G/P \tag{2.2.17}$$

are fiber bundles with fiber isomorphic to P/B, which is a homogeneous variety as well (see for example the notes (Ott95, Page 55) and the references therein). In particular, if we choose P such that π is an extremal contraction, it follows that π is a $\mathbb{P}^1$-bundle. This tells us that a complete G-flag variety has r distinct $\mathbb{P}^1$-bundle structures, where r is the rank of G.

2.2.3 Example: Grassmannians and flags

As an example, let us work out the case of $\mathfrak{g}$ =$\mathbf{sl}(n, \mathbb{C})$, the algebra of traceless $n \times n$ matrices, corresponding to the group $G = SL(n, \mathbb{C})$. We will find parabolic subalgebras giving rise to the partial flag varieties of $\mathbb{C}^n$.

Let us choose a basis for $\mathfrak{h}$ given by matrices h_i with components $(h_i)_{pq} = \delta_{ip}\delta_{iq} - \delta_{i+1p}\delta_{i+1q}$ for $1 \leq i \leq n-1$ (all entries are zero except for two consecutive entries on the diagonal, which are 1 and -1) and let us complete it to a basis of $\mathfrak{g}$ by adding the matrices m_{ij} with components $(m_{ijp}q) = \delta_{ip}\delta_{jq}$ for $1 \leq i \neq j \leq n$ (i.e. all entries are zero but the entry ij, which is equal to one). If we now call $h = \mathrm{diag}(t_1, \ldots, t_n)$ we have

$$[h, m_{ij}] = (t_i - t_j)m_{ij}$$

and this defines the roots expanded on the standard basis of $M_{n\times n}$, in fact, given an element h_i of the standard basis of $\mathfrak{h}$, by adapting Equation 2.2.2 to our double-index notation we write:

$$[h_i, m_{jk}] = (\delta_{ij} - \delta_{ik})m_{jk} = \alpha_{jk}(h_i)m_{jk}.$$

Observe that $\alpha_{jk} = -\alpha_{kj}$.

For simplicity of notation, let us fix $n = 5$. Then, let us consider the following relation:

$$(\alpha_{12} + \alpha_{23})(h_i) = \delta_{i1} - \delta_{i2} + \delta_{i2} - \delta_{i3} = \alpha_{13}(h_i).$$

The same reasoning gives:

$$\alpha_{ij} + \alpha_{jk} = \alpha_{ik}$$

for every i, j, k such that $i < j < k \leq n$. so, if we choose $\mathcal{S} = \{\alpha_{12}, \alpha_{23}, \alpha_{34}, \alpha_{45}\}$ the positive roots which are not simple correspond to off-diagonal entries above the first upper-diagonal line. This is made clearer by the following picture:

$$\begin{pmatrix} & \alpha_{12} & \bullet & \bullet & \bullet \\ & & \alpha_{23} & \bullet & \bullet \\ & & & \alpha_{34} & \bullet \\ & & & & \alpha_{45} \\ & & & & \end{pmatrix}$$

where the dots represent positive roots which are not simple roots. Similarly, the negative roots correspond to entries below the diagonal.

We can now characterize the Borel subalgebra: By Equation 2.2.14 we find that $\mathfrak{b}$ is the algebra of upper triangular traceless matrices, because the direct sum of the root spaces corresponding to positive roots is the subspace of strictly upper triangular matrices. Before proceeding with the characterization of parabolic subgroups, let us discuss the geometry related to the group B. Exponentiating $\mathfrak{b}$ we obtain

$$B = \{b \in SL(n), \text{upper triangular}\}$$

Then, B clearly acts on $SL(n)$ by $g, b \longrightarrow gb^{-1}$ where the inverse is to preserve associativity (the choice of right inverse multiplication insetead of left multiplication is for further convenience). This action sends the i-th column of g to a linear combination of the first i columns. Clearly, an equivalence class is a chain of subspace of dimensions ranging from 1 to 5. Hence, with respect to this action, G/B is the variety of complete flags $F(1, 2, 3, 4, 5)$.

Now let us construct the parabolic subalgebra $\mathfrak{p}_{23}$. The first step is to construct the Levi subalgebra: By Equation 2.2.15 we learn that $\mathfrak{l}_{23}$ is the direct sum of $\mathfrak{h}$ with the span of the first and the fourth simple roots. We get:

$$\mathfrak{l}_{23} = \begin{pmatrix} \bullet & \bullet & & & \\ \bullet & \bullet & & & \\ & & \bullet & & \\ & & & \bullet & \bullet \\ & & & \bullet & \bullet \end{pmatrix}$$

where the bullets correspond to the only entries which are allowed to be nonzero. To get $\mathfrak{p}_{23}$ we need to add up the nilpotent subalgebra, which is given by the sum of the root spaces corresponding to positive root which are not appearing in Δ_{23} Pictorially, adding $\mathfrak{n}_{23}$ correspond to "filling up the upper diagonal part". In other words, we get:

$$\mathfrak{p}_{23} = \begin{pmatrix} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ & & \bullet & \bullet & \bullet \\ & & & \bullet & \bullet \\ & & & \bullet & \bullet \end{pmatrix}$$

Clearly, the action of matrices of this shape on elements of $SL(5)$ preserves the pairs of nested subspaces given respectively by the span of the first two and the first three columns. Therefore, the quotient of $SL(5)$ by this action is isomorphic to the flag variety $F(2,3,5)$.

2.3 Homogeneous vector bundles

Let G be a linear reductive group and $P \subset G$ a parabolic subrgoup. Homogeneous vector bundles form a very important class of vector bundles over G/P determined by the representations of P. Lie theory provides an extremely useful tool to compute their cohomology, in the form of Borel–Weil–Bott's theorem.

Given a linear reductive group G, a parabolic subgroup P and a representation $\Gamma : P \longrightarrow \mathrm{End}(V)$, we will use the notation $G \times^P V$ to denote the *balanced product* (see, for example, the notes (Mit01, Section 3)), i.e. following:

$$G \times^P V := G \times V/\sim \tag{2.3.1}$$

where the equivalence relation is given by $(g, v) \sim (gp^{-1}, \Gamma(p)v)$.

Definition 2.3.1. *Let G/P be a smooth homogeneous variety and Γ_λ : $P \longrightarrow \mathrm{Aut}(V_\lambda^{(P)})$ a representation of P with highest weight λ. We call* homogeneous vector bundle *the quasiprojective variety $\mathcal{E}_\lambda$ given by the following construction:*

$$\begin{tikzcd} \mathcal{E}_\lambda = G \times^P V_\lambda^{(P)} \arrow[d, "\pi"] \\ G/P \end{tikzcd} \tag{2.3.2}$$

This is indeed a vector bundle of rank $\dim V_\lambda^{(P)}$ over G/P. Sections $s \in H^0(G/P, \mathcal{E})$ are in one to one correspondence with equivariant maps:

$$\hat{s} : G \longrightarrow V_\lambda^{(P)} \tag{2.3.3}$$

satisfying $s([g]) = [g, \hat{s}(g)]$ for every $g \in G$ (here $[-]$ denote an equivalence class under the action of P). The equivariancy condition, for every $g \in G$ and $p \in P$, is the following:

$$s(p.g) = (pg, \Gamma_\lambda(p)\hat{s}(g)) \tag{2.3.4}$$

Definition 2.3.2. *We say that a homogeneous vector bundle $\mathcal{E}_\lambda$ is* irreducible *if the associated representation Γ_λ of P is irreducible.*

2.3.1 Example: the universal sequence on $G(2,5)$

The Grassmannian $G(2,5)$ is given by the quotient of $SL(5)$ by the parabolic subgroup associated to the second fundamental weight. By the discussion of Section 2.2.2, such parabolic subgroup has the following description:

$$P_2 = \left\{ \begin{pmatrix} tSL(2) & N \\ 0 & t^{-1}SL(3) \end{pmatrix} \right\} \tag{2.3.5}$$

where $t \in \mathbb{C}^*$ and N is the nonzero component of a nilpotent factor. The fundamental representation of $SL(5)$ is given by the following map:

$$\begin{aligned} \Gamma_{\omega_1} : SL(V) &\longrightarrow \mathrm{Aut}(V_{\omega_1}^{(SL(5))}) \\ g &\longmapsto \{\Gamma_{\omega_1}(g) : v \mapsto gv\} \end{aligned} \tag{2.3.6}$$

where $V_{\omega_1}^{(SL(5))} \simeq V$. If we restrict this representation to P_2, we obtain a trivial vector bundle over $G(2,V)$:

$$V \otimes \mathcal{O} = SL(V) \times^{P_2} V_{\omega_1}^{(SL(5))} \tag{2.3.7}$$

If we consider the action of P_2 on the subspace $W \subset V_{\omega_1}^{(SL(5))}$ given by

$$W = \left\{ v \in V_{\omega_1} : v = (v_1, v_2, 0, 0, 0) \right\}$$

we see that the $SL(2)$-block of P acts on the nonzero component of an element of W by left multiplication. In particular, $W \subset V$ is closed under P_2. This allows us to define a rank 2 subbundle of $V \otimes \mathcal{O}$, which is the *tautological* or *universal* bundle of $G(2, V)$:

$$\mathcal{U} = SL(V) \times^{P_2} W \subset V \otimes \mathcal{O} \tag{2.3.8}$$

A similar explicit description of the quotient bundle

$$\mathcal{Q} = V \otimes \mathcal{O} / \mathcal{U}$$

can be realized taking the vector space $Z \simeq \mathbb{C}^3$ which is the image of the following surjection of vector spaces:

$$\begin{aligned} \pi : V &\longrightarrow Z \\ v &\longmapsto (0, 0, v_3, v_4, v_5) \end{aligned} \tag{2.3.9}$$

This surjection is equivariant, hence it defines a surjection of vector bundles. We obtain the tautological exact sequence for $G(2, V)$:

$$0 \longrightarrow \mathcal{U} \longrightarrow V \otimes \mathcal{O} \longrightarrow \mathcal{Q} \longrightarrow 0. \tag{2.3.10}$$

The same reasoning gives rise to tautological exact sequences on every Grassmannian. Moreover, pullbacks of tautological bundles define similar sequences on flag varieties. For instance, given a flag variety $F(k, l, m)$ one has the following short exact sequence:

$$0 \longrightarrow \mathcal{U}_{G(k,m)} \longrightarrow \mathcal{U}_{G(l,m)} \longrightarrow \mathcal{P} \longrightarrow 0 \tag{2.3.11}$$

where $\mathcal{U}_{G(k,m)}$ and $\mathcal{U}_{G(l,m)}$ are the pullbacks of the tautological bundles of respectively $G(k, m)$ and $G(l, m)$, and $\mathcal{P}$ is a rank $l - k$ quotient bundle.

2.4 Cohomology of vector bundles on generalized flag varieties: the Borel–Weil–Bott theorem

Throughout this work, we will mostly deal with homogeneous vector bundles and their cohomology. The problem of computing such cohomology is completely solved by the Borel–Weil–Bott theorem. However, applying such theorem often leads to cumbersome calculations, whose difficulty increases with the complexity of the automorphism group of the variety. The goal of the following section is to establish a comfortable notation in order to apply the Borel–Weil–Bott theorem to any homogeneous vector bundle on a homogeneous variety, in a simple algorithmic way. This algorithm is based on Weyl reflections, therefore we start from a shorthand notation which applies to every semisimple Lie algebra. Let us first introduce a uniformized notation for homogeneous varieties:

Definition 2.4.1. *Let G be a semisimple Lie group of rank r. We call* G-flag variety *any homogeneous variety $X = G/P$ where $P \subset G$ is a parabolic subgroup. We say that a G-flag variety is a* G-Grassmannian *if it has Picard number one. We call* complete G-flag variety *the quotient G/B.*

Let G be a semisimple Lie group of rank r and $B \subset G$ its Borel subgroup. Then, as we discussed in Remark 2.2.11, there exist projections $G/B \longrightarrow G/P$ from the complete G-flag variety. Let us summarize here some results on vector bundles on G/B and their pushforwards to the other G-flags.

Lemma 2.4.2. *(Wey03, Proposition 4.1.3) Every line bundle $\mathcal{L}$ on a complete G-flag has the form*

$$\mathcal{L} \simeq \mathcal{E}_\omega \tag{2.4.1}$$

for some weight ω.

Theorem 2.4.3 (Borel–Weil–Bott for line bundles)**.** *Let G be a semisimple Lie group and $B \subset G$ a Borel subgroup. Let λ be an integral weight over G/B and $\mathcal{E}_\lambda$ the associated line bundle. Call ρ the sum of all fundamental weights. Then one and only one of the following situations occur:*

- *There exists a nontrivial Weyl reflection S such that $S(\lambda+\rho)-\rho = \lambda$. Then $H^\bullet(G/B, \mathcal{E}_\lambda) = 0$.*
- *There exists a unique Weyl reflection S such that $S(\lambda)$ is a dominant integral weight. Then $H^\bullet(G/B, \mathcal{E}_\lambda) = V_{S(\lambda+\rho)-\rho}[-l(S)]$.*

The following lemma allows to use Theorem 2.4.3 to compute the cohomology of irreducible homogeneous vector bundles on any homogeneous variety, and leads to the second formulation of Borel–Weil–Bott's theorem (Theorem 2.4.5 in the following).

Lemma 2.4.4. *Every irreducible homogeneous vector bundle $\mathcal{F}$ on a G-flag variety G/P has the form*

$$\mathcal{F} \simeq \pi_* \mathcal{E}_\lambda \tag{2.4.2}$$

where $\mathcal{E}_\lambda$ is a homogeneous line bundle for some weight ω, and $\pi : G/B \longrightarrow G/P$.

Proof. For the sake of self-containedness of this exposition, we will give a proof of this lemma, despite it is a well-known result. First, let us observe that for every $x \in G/P$ one has $\pi^{-1}(x) \simeq P/B$. Fix a P-dominant weight λ. By Theorem 2.4.3, P-dominance of λ implies that $H^0(G/B, \mathcal{E}_\lambda)$ is the representation space of the representation of P associated to the highest weight λ, let us call such representation space V_λ. In fact, one has $\mathcal{E}_\lambda \simeq P \times^B \mathbb{C}$ where the B-action on $\mathbb{C}$ is given by the character of weight λ. Then, by Leray spectral sequence, $H^0(\pi^{-1}(x), \mathcal{E}_\lambda|_{\pi^{-1}}) = V_\lambda$. Then, we can construct a homogeneous vector bundle $\mathcal{F}_\lambda := G \times^P V_\lambda$ over G/P and we see that $\pi_* \mathcal{E}_\lambda = \mathcal{F}_\lambda$. On the other hand, every irreducible homogeneous vector bundle on G/P has the form $G \times^P E_0$ hence there exists a line bundle $\mathcal{E}_\lambda$ such that $\pi_* \mathcal{E}_\lambda = \mathcal{F}_\lambda$ where λ is a P-dominant weight. □

Theorem 2.4.5 (Borel–Weil–Bott for vector bundles)**.** *Let G be a semisimple Lie group and $P \subset G$ a parabolic subgroup. Let λ be an integral weight over G/B and $\mathcal{E}_\lambda$ the associated vector bundle. Call ρ the sum of all fundamental weights. Then one and only one of the following situations occur:*

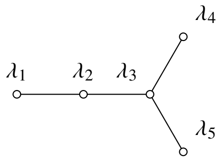
- *There exists a nontrivial Weyl reflection S such that $S(\lambda+\rho)-\rho = \lambda$. Then $H^\bullet(G/P, \mathcal{E}_\lambda) = 0$.*
- *There exists a unique Weyl reflection S such that $S(\lambda)$ is a dominant integral weight. Then $H^\bullet(G/P, \mathcal{E}_\lambda) = V_{S(\lambda+\rho)-\rho}[-l(S)]$.*

Remark 2.4.6. A very useful consequence of this result is that irreducible homogeneous vector bundles on any homogeneous variety G/P have nonvanishing cohomology in at most one degree.

Weyl reflections on the Dynkin diagram

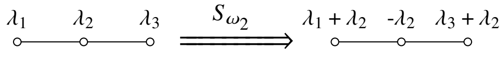
In this section we will describe a simple method to compute cohomology of any irreducible homogeneous vector bundle on a homogeneous variety, given the weight of the associated representation.

Let us consider a highest weight $\lambda = \lambda_1\omega_1 + \cdots + \lambda_n\omega_n$ on a rank n semisimple Lie algebra. Let us write the weight directly on the Dynkin diagram in the following way:

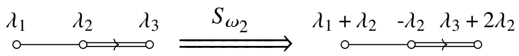


The action of Weyl reflection is described by Equation 2.2.7. Since Dynkin diagrams are a graphical way to express the data contained in the Cartan matrix, which is the data defining Weyl reflections as well, we can write a set of simple rules which tell us how to perform a given Weyl reflection, simply by reading the Dynkin diagram. Since we are writing the highest weight on the Dynkin diagram, we can talk about a Weyl reflection respect to a *node* referring to the reflection associated to the fundamental weight which corresponds to that node. The following rules can be deduced simply by using Equation 2.2.7 to perform the computations explicitly.

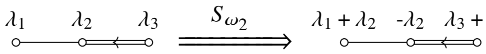
- Weyl reflection with respect to a node connected by simple lines:



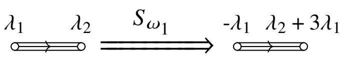
- Weyl reflection with respect to a node connected by an outward-directed double line:



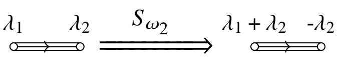
- Weyl reflection with respect to a node connected by an inward-directed double line:



- Weyl reflection with respect to a node connected by an outward-directed triple line:



- Weyl reflection with respect to a node connected by an inward-directed triple line:

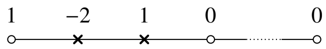


Example 2.4.7. Let us consider the flag variety $F(2,3,n)$ and the projections p and q to its Grassmannians $G(2,n)$ and $G(3,n)$. We get the following diagram:

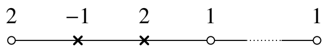
$$\begin{tikzcd} & F(2,3,n) \arrow[dl, "p"'] \arrow[dr, "q"] & \\ G(2,n) & & G(3,n) \end{tikzcd} \tag{2.4.3}$$

Let us call $\mathcal{U}$ the tautological bundle of $G(2,n)$. It is a homogeneous vector bundle of rank 2. With the notation $\mathcal{O}(a,b) = p^*\mathcal{O}(a)\otimes q^*\mathcal{O}(b)$, we illustrate the method above computing $H^\bullet(F(2,3,n), p^*\mathcal{U}^\vee(-2,1))$. First, the flag variety $F(2,3,n)$ is a $GL(n)$-homogeneous variety described as $GL(n)/P^{2,3}$. The associated Dynkin diagram is:

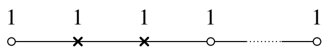
The weight associated to $p^*\mathcal{U}^\vee(-2,1)$ is $\omega = \omega_1 - 2\omega_2 + \omega_3$. We can write it on the Dyinkin diagram in the following way:



In order to apply the Borel–Weil–Bott algorithm, we first need to add to ω the sum of fundamental weights, obtaining the following:



We can now start with Weyl reflections. Since the second coefficient of our weight is negative, we apply S_{ω_2} and we get:



This last weight is dominant, hence we can subtract back the sum of fundamental weights obtaining the trivial weight $(0, \dots, 0)$ corresponding to the trivial representation of dimension 1. Since we used only one Weyl reflection, the cohomology is concentrated in degree one, therefore we conclude that:

$$H^k(F(2,3,n), p^*\mathcal{U}^\vee(-2,1)) = \begin{cases} \mathbb{C} & k = 1 \\ 0 & k \neq 1 \end{cases} \tag{2.4.4}$$

Remark 2.4.8. A geometric interpretation of the result we got from Example 2.4.7 is the following: since $H^\bullet(F(2,3,n), p^*\mathcal{U}^\vee(-2,1)) = \mathrm{Ext}^\bullet((\mathcal{O}(1,-1), p^*\mathcal{U}^\vee(-1,0))$, the outcome of our computation tells that there exists a unique extension between $p^*\mathcal{U}^\vee(-1,0)$ and $\mathcal{O}(1,-1)$. By the isomorphism $\mathcal{U} \simeq \mathcal{U}^\vee(-1)$ we associate such extension to the pullback $q^*\widetilde{\mathcal{U}}$ of the tautological bundle of $G(3,n)$, i.e. to the sequence:

$$0 \longrightarrow p^*\mathcal{U} \longrightarrow q^*\widetilde{\mathcal{U}} \longrightarrow \mathcal{O}(1,-1) \longrightarrow 0 \qquad (2.4.5)$$

2.5 Calabi–Yau zero loci of homogeneous vector bundles

In the vast literature on Calabi–Yau varieties, there exist several overlapping definitions with different degrees of strength. Therefore, let us begin by fixing the Calabi–Yau conditions that will be used through all the remainder of this work.

Definition 2.5.1. *A* Calabi–Yau variety *is an algebraic variety X such that $\omega_X \simeq \mathcal{O}_X$ and $H^m(X,\mathcal{O}_X) = 0$ for $0 < m < \dim(X)$. We call* Calabi–Yau fibration *a fibration $X \longrightarrow B$ such that the general fiber is a Calabi–Yau variety.*

Remark 2.5.2. In the following chapters, we will encounter Calabi–Yau fibrations such that their total space is itself a Calabi–Yau variety. To avoid the potential confusion that such notion could arise, we will clearly refer to them as Calabi–Yau varieties with a Calabi–Yau fibration structure.

Lemma 2.5.3. *Let $\mathcal{E}$ be a rank r homogeneous vector bundle on an n-dimensional homogeneous variety $Z = G/P$ and call h its structure morphism. Let $X = \mathbb{P}(\mathcal{E})$ and call $\mathcal{O}(1)$ the associated Grothendieck line bundle, assume that $\mathcal{O}(1)$ is ample. Then, if $\det(\mathcal{E}) \otimes \omega_Z \simeq \mathcal{O}$, the zero locus Y of a general section of $\mathcal{E}$ is either empty or a Calabi–Yau variety of dimension $n - r$.*

Proof. Call $H = H^0(X, \mathcal{O}(1))$. and fix a general section $\sigma \in H$ such that $Y = Z(h_*\sigma))$. Since $\mathcal{O}(1)$ is an ample line bundle, $\mathcal{E}$ is an ample vector bundle. $\mathcal{O}(1)$ is an homogeneous ample line bundle, hence it is globally generated. We have the following sequence on X:

$$0 \longrightarrow K \longrightarrow H \otimes \mathcal{O} \longrightarrow \mathcal{O}(1) \longrightarrow 0 \tag{2.5.1}$$

By applying the derived pushforward functor, which is left exact, to the sequence 2.5.1 we conclude that $\mathcal{E}$ is globally generated if $R^1h_*K = 0$. But this is true because $H^0(X, K) \simeq H^1(X, K)$ and since K is homogeneous, $H^0(X, K) = H^1(X, K) = 0$. Note that K is flat over Z and the map $z \longmapsto \dim H^1(h^{-1}(z), K|_{h^{-1}(z)})$ is constant: by (Mum12, Page 50, Corollary 2), we get $R^1h_*K_z \simeq H^1(h^{-1}(z), K|_{h^{-1}(z)})$. On every fiber $h^{-1}(z)$ the sequence 2.5.1 restricts to the Euler sequence of $h^{-1}(z)$ and this proves that $H^1(h^{-1}(z), K|_{h^{-1}(z)}) = 0$ for every z, hence R^1h_*K vanishes on every stalk. Thus Y is of expected codimension by generality of σ, in fact, $h_*\sigma$ is general if σ is general. If $n - r \leq 0$ there is nothing more to prove. Otherwise, let us proceed in the following way: by assumption, and by adjunction formula, Y has vanishing first Chern class. By (Laz04b, Example 7.1.5), since $\mathcal{E}$ is ample, the restriction map $H^q(Z, \Omega^p_Z) \longrightarrow H^q(Y, \Omega^p_Y)$ is an isomorphism for $p+q < \dim(Y)$, in particular $H^q(Z, \mathcal{O}_Z) \simeq H^q(Y, \mathcal{O}_Y)$ for $q < \dim(Y)$. But since Z is homogeneous $H^\bullet(Z, \mathcal{O}_Z) \simeq \mathbb{C}[0]$ and this concludes the proof. Alternatively, one can deduce smoothness of Y_i by smoothness of $Z(\sigma)$ and (DK20, Lemma 3.2). $\square$

3 Derived categories

Derived categories of coherent sheaves, in light of homological mirror symmetry conjectures, quickly became a major research topic in algebraic geometry and mathematical physics. The reconstruction theorem of Bondal and Orlov (BO01) allows to determine whether two Fano or general type varieties are isomorphic by studying their derived categories, while the same does not occur for varieties with trivial canonical class: a remarkable example of non birationally equivalent, derived equivalent Calabi–Yau threefolds has been found by Borisov and Caldararu (BC08) in terms of the Pfaffian–Grassmannian pair. Many other examples of pairs of non isomorphic (or even non birational), derived equivalent Calabi–Yau varieties have been found in the following years. In this chapter, we will review some basic tools of manipulating semiorthogonal decompositions and exceptional collections, which will serve as the main tools for Chapters 9, 10 and 11.

3.1 Semiorthogonal decompositions

Hereafter, following (Orl03), we collect some introductory material on derived categories.

Definition 3.1.1. *(Orl03, Definition 2.2.2) Let $\mathfrak{C}$ be a triangulated category and $I : \mathfrak{A} \hookrightarrow \mathfrak{C}$ an embedding of a full triangulated subcategory. We say that $\mathfrak{A}$ is* right admissible *if I has a right adjoint $I^! : \mathfrak{C} \longrightarrow \mathfrak{A}$. Similarly, we call $\mathfrak{A}$* left admissible *if I has a left adjoint $I^* : \mathfrak{C} \longrightarrow \mathfrak{A}$.*

Definition 3.1.2. *(Orl03, Definition 2.2.1) Let* $\mathfrak{C}$ *be an additive category and* $\mathfrak{A} \subset \mathfrak{C}$ *a full additive subcategory. We call* right orthogonal *to* $\mathfrak{A}$ *in* $\mathfrak{C}$ *the following full subcategory of* $\mathfrak{C}$*:*

$$\mathfrak{A}^{\perp} := \{\mathcal{F} \in \mathfrak{C} : \mathrm{Ext}^{\bullet}(\mathcal{E}, \mathcal{F}) = 0 \textit{ for every } \mathcal{E} \in \mathfrak{A}\} \tag{3.1.1}$$

In the same way, we define left orthogonal *to* $\mathfrak{A}$ *in* $\mathfrak{C}$ *the following full subcategory of* $\mathfrak{C}$*:*

$$^{\perp}\mathfrak{A} := \{\mathcal{F} \in \mathfrak{C} : \mathrm{Ext}^{\bullet}(\mathcal{F}, \mathcal{E}) = 0 \textit{ for every } \mathcal{E} \in \mathfrak{A}\} \tag{3.1.2}$$

In order to ensure that the (left or right) semiorthogonal complement of an admissible subcategory is admissible, we need to add the requirement of *saturatedness* to the category $\mathfrak{C}$, i.e. we require that every exact functor $\mathfrak{C} \longrightarrow D^b(\mathbb{C})$ is representable (BK).

Saturatedness will always be satisfied when needed, since the derived category of coherent sheaves of a smooth projective variety is always saturated (BV03).

Definition 3.1.3. *(Orl03, Definition 2.2.3) Let* $\mathfrak{C}$ *be a saturated triangulated category. Then, we call* semiorthogonal decomposition *of* $\mathfrak{C}$ *a sequence of full triangulated admissible subcategories* $\{\mathfrak{A}_1, \dots, \mathfrak{A}_N\}$ *such that:*

- *the smallest full subcategory of* $\mathfrak{C}$ *containing* $\mathfrak{A}_1, \dots, \mathfrak{A}_N$ *coincides with* $\mathfrak{C}$
- *one has* $\mathrm{Ext}^{\bullet}(\mathcal{E}, \mathcal{F}) = 0$ *if* $\mathcal{E} \in \mathfrak{A}_i$*,* $\mathcal{F} \in \mathfrak{A}_j$ *and* $i > j$.

Following the usual notation, we represent a semiorthogonal decomposition in the following way:

$$\mathfrak{C} = \langle \mathfrak{A}_1, \dots, \mathfrak{A}_N \rangle. \tag{3.1.3}$$

Definition 3.1.4. *(Kuz07, Definition 4.1) Let X be a smooth projective algebraic variety with a line bundle $\mathcal{O}(1)$. We call* Lefschetz semiorthogonal decomposition *the following expression:*

$$D^b(X) = \langle \mathfrak{A}_0, \mathfrak{A}_1(1) \dots, \mathfrak{A}_{m-1}(m-1) \rangle \tag{3.1.4}$$

where one has $\mathfrak{A}_{m-1} \subseteq \mathfrak{A}_{m-2} \subseteq \cdots \subseteq \mathfrak{A}_0$. The integer m is called length *or* index *of the Lefschetz decomposition. A Lefschetz decomposition is called* rectangular *if $\mathfrak{A}_{m-1} = \mathfrak{A}_{m-2} = \cdots = \mathfrak{A}_0$.*

Definition 3.1.5. *Let $\mathfrak{C}$ be a triangulated category. We say that an object $\mathcal{E} \in \mathfrak{A}$ is* exceptional *if it satisfies the following cohomological conditions:*

- $\mathrm{Ext}^i(\mathcal{E}, \mathcal{E}) = 0$ *for $i \neq 0$*
- $\mathrm{Hom}(\mathcal{E}, \mathcal{E}) = 0$

Definition 3.1.6. *Let $\mathfrak{C}$ be a saturated triangulated category. We call* exceptional collection *a set of exceptional objects $\{\mathcal{E}_1, \dots, \mathcal{E}_N\}$ such that* $\mathrm{Ext}^\bullet(\mathcal{E}_i, \mathcal{E}_j) = 0$ *for $i > j$. Moreover, if $\{\mathcal{E}_1, \dots, \mathcal{E}_N\}$ generate $\mathfrak{C}$, we say that they form a* full exceptional collection*, which will be denoted in the following way:*

$$\mathfrak{C} = \langle \mathcal{E}_1, \dots, \mathcal{E}_N \rangle. \tag{3.1.5}$$

In other words, a full exceptional collection is a semiorthogonal decomposition such that every block is a subcategory generated by a single exceptional object.

Definition 3.1.7. *(Kuz07, Section 2.7) Let us consider two algebraic varieties X, Y with morphisms $f : X \longrightarrow S$, $g : Y \longrightarrow S$ to a smooth variety S A functor $F : D^b(X) \longrightarrow D^b(Y)$ is called S-*linear *if for every $\mathcal{E} \in D^b(X)$, $\mathcal{F} \in D^b(S)$ one has:*

$$F(\mathcal{E} \otimes f^*\mathcal{F}) \simeq F(\mathcal{E}) \otimes g^*\mathcal{F} \tag{3.1.6}$$

Moreover, a strictly full subcategory $\mathfrak{C} \subset D^b(X)$ is called S-linear if for all $\mathcal{E} \in \mathfrak{C}$ and $\mathcal{F} \in D^b(S)$ one has $f^*\mathcal{F} \otimes \mathcal{E} \in \mathfrak{C}$.

3.2 Mutations

Derived categories of coherent sheaves are closed under the operations of taking cones, direct sums and shifts. This allows to construct transformations of full exceptional collections called *mutations*, which are extremely useful in applications. In the following exposition, where not differently stated, we refer to (Kuz07). Given a saturated triangulated category $\mathfrak{C}$, let us consider the following:

$$\mathfrak{C} = \langle \mathfrak{A}, {}^\perp \mathfrak{A} \rangle, \quad \mathfrak{C} = \langle \mathfrak{A}^\perp, \mathfrak{A} \rangle \tag{3.2.1}$$

Both semiorthogonal decompositions are well defined by definition of semiorthogonal complement. Therefore, by (Bon89, Lemma 2.3), ${}^\perp\mathfrak{A}$ and $\mathfrak{A}^\perp$ are admissible subcategories.

Definition 3.2.1. *Let $\mathfrak{C}$ be a triangulated subcategory and $\mathfrak{A} \subset \mathfrak{C}$ an admissible subcategory. Let us consider the fully faithful functors $i_{{}^\perp\mathfrak{A}} : {}^\perp\mathfrak{A} \hookrightarrow \mathfrak{C}$ and $i_{\mathfrak{A}^\perp} : \mathfrak{A}^\perp \hookrightarrow \mathfrak{C}$. We call* left *and* right mutation *with respect to $\mathfrak{A}$ the following functors:*

$$\mathbb{L}_{\mathfrak{A}} := i_{\mathfrak{A}^\perp} i^*_{\mathfrak{A}^\perp}, \quad \mathbb{R}_{\mathfrak{A}} := i_{{}^\perp\mathfrak{A}} i^!_{{}^\perp\mathfrak{A}} \tag{3.2.2}$$

One has the following standard facts (see for example (Bon89; BO02)):

Proposition 3.2.2. *Let $\mathfrak{C}$ be a saturated triangulated category and, for $n \leq 2$, let $\mathfrak{B}$ and $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ be admissible subcategories. Then, the following holds:*

- *For every object $\mathcal{E} \in \mathfrak{C}$ there exist the following distinguished triangles:*

$$\begin{gathered} i_{\mathfrak{B}} i^!_{\mathfrak{B}} \mathcal{E} \longrightarrow \mathcal{E} \longrightarrow \mathbb{L}_{\mathfrak{B}} \mathcal{E} \xrightarrow{[1]} \\ \mathbb{R}_{\mathfrak{B}} \mathcal{E} \longrightarrow \mathcal{E} \longrightarrow i_{\mathfrak{B}} i^*_{\mathfrak{B}} \mathcal{E} \xrightarrow{[1]} \end{gathered} \tag{3.2.3}$$

- *The restrictions $\mathbb{L}_{\mathfrak{B}}|_{\mathfrak{B}}$ and $\mathbb{R}_{\mathfrak{B}}|_{\mathfrak{B}}$ are the zero functors.*

- *The restrictions $\mathbb{L}_{\mathfrak{B}}|_{\mathfrak{B}^\perp}$ and $\mathbb{R}_{\mathfrak{B}}|_{{}^\perp\mathfrak{B}}$ are the identity functors.*

- *The functors $\mathbb{L}_{\mathfrak{B}}|_{{}^\perp\mathfrak{B}}$ and $\mathbb{R}_{\mathfrak{B}}|_{\mathfrak{B}^\perp}$ are mutually inverse.*

- *If $\mathfrak{C}$ has a Serre functor $\mathcal{S}_{\mathfrak{C}}$, then $\mathbb{L}_{\mathfrak{B}}|_{{}^\perp\mathfrak{B}} = \mathcal{S}_{\mathfrak{C}}|_{{}^\perp\mathfrak{B}}$ and $\mathbb{R}_{\mathfrak{B}}|_{\mathfrak{B}^\perp} = \mathcal{S}^{-1}_{\mathfrak{C}}|_{\mathfrak{B}^\perp}$.*

- *If $\mathfrak{C} = \langle \mathfrak{A}_1, \ldots, \mathfrak{A}_n \rangle$ one has:*

$$\begin{aligned} \langle \mathfrak{A}_1, \ldots, \mathfrak{A}_n \rangle &= \langle \mathfrak{A}_1, \ldots, \mathbb{L}_{\mathfrak{A}_{k-1}} \mathfrak{A}_k, \mathfrak{A}_{k-1}, \mathfrak{A}_{k+1}, \ldots \mathfrak{A}_n \rangle \\ &= \langle \mathfrak{A}_1, \ldots, \mathfrak{A}_{k-1}, \mathfrak{A}_{k+1}, \mathbb{R}_{\mathfrak{A}_{k+1}} \mathfrak{A}_k, \ldots \mathfrak{A}_n \rangle. \end{aligned} \tag{3.2.4}$$

The easiest example is given by mutations of pairs of exceptional objects inside a full exceptional collection. For the sake of simplicity, let us consider the case where our category $\mathfrak{C}$ is generated by two semiorthogonal exceptional objects:

Example 3.2.3 (Mutations of pairs). Let $\mathcal{E}_1, \mathcal{E}_2$ be exceptional objects such that $\mathrm{Ext}^\bullet(\mathcal{E}_2, \mathcal{E}_1) = 0$ and consider the category $\langle \mathcal{E}_1, \mathcal{E}_2 \rangle$ generated by them. For $j \in \{1; 2\}$ we have the fully faithful functors:

$$\langle \mathcal{E}_j \rangle \xhookrightarrow{i_j} \langle \mathcal{E}_1, \mathcal{E}_2 \rangle \tag{3.2.5}$$

which have, respectively, the following left and right adjoints (see, for example, the notes (Shi) and the references therein):

$$\begin{aligned} \langle \mathcal{E}_1, \mathcal{E}_2 \rangle &\xrightarrow{i_j^!} \langle \mathcal{E}_j \rangle \\ \mathcal{F} &\longmapsto \mathcal{E}_j \otimes \mathrm{Ext}^\bullet(\mathcal{E}_j, \mathcal{F}) \end{aligned} \tag{3.2.6}$$

$$\begin{aligned} \langle \mathcal{E}_1, \mathcal{E}_2 \rangle &\xrightarrow{i_j^*} \langle \mathcal{E}_j \rangle \\ \mathcal{F} &\longmapsto \mathcal{E}_j \otimes \mathrm{Ext}^\bullet(\mathcal{F}, \mathcal{E}_j)^\vee \end{aligned} \tag{3.2.7}$$

Therefore, by Proposition 3.2.2, one has the following triangles in $\langle \mathcal{E}_1, \mathcal{E}_2 \rangle$:

$$\begin{gathered} \mathcal{E}_1 \otimes \mathrm{Ext}^\bullet(\mathcal{E}_1, \mathcal{E}_2) \longrightarrow \mathcal{E}_2 \longrightarrow \mathbb{L}_{\mathcal{E}_1}\mathcal{E}_2 \xrightarrow{[1]} \\ \mathbb{R}_{\mathcal{E}_2}\mathcal{E}_1 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E}_2 \otimes \mathrm{Ext}^\bullet(\mathcal{E}_1, \mathcal{E}_2)^\vee \xrightarrow{[1]} \end{gathered} \tag{3.2.8}$$

and this leads to the following expressions for mutations of pairs:

$$\begin{aligned} \mathbb{L}_{\mathcal{E}_1}\mathcal{E}_2 &= \mathrm{Cone}(\mathcal{E}_1 \otimes \mathrm{Ext}^\bullet(\mathcal{E}_1, \mathcal{E}_2) \longrightarrow \mathcal{E}_2) \\ \mathbb{R}_{\mathcal{E}_2}\mathcal{E}_1 &= \mathrm{Cone}(\mathcal{E}_1 \longrightarrow \mathcal{E}_2 \otimes \mathrm{Ext}^\bullet(\mathcal{E}_1, \mathcal{E}_2)^\vee)[-1] \end{aligned} \tag{3.2.9}$$

3.3 Semiorthogonal decompositions and fibrations

In this section we will review several known results about semirothogonal decompositions of derived categories of different kinds of fibrations.

3.3.1 Semiorthogonal decomposition of a projective bundle

Let B be a smooth projective variety and $\mathcal{E} \longrightarrow B$ a vector bundle of rank r. Let us fix $X = \mathbb{P}(\mathcal{E})$ with the structure morphism $\pi : X \longrightarrow B$, call L the associated Grothendieck line bundle, with a surjection

$$\mathcal{E} \longrightarrow L \longrightarrow 0 \tag{3.3.1}$$

given by the relative Euler sequence. Then, the following result gives a semiorthogonal decomposition for X.

Theorem 3.3.1. *(Orl92, Corollary 2.7) Let X, E and L be as above. Then the functor $\pi^* : D^b(B) \longrightarrow D^b(X)$ is fully faithful, and there exists the following Lefschetz semiorthogonal decomposition:*

$$D^b(X) = \langle \pi^* D^b(B), \pi^* D^b(B) \otimes L, \ldots, \pi^* D^b(B) \otimes L^{\otimes(r-1)} \rangle. \tag{3.3.2}$$

There exists a useful generalization of Theorem 3.3.1 to flat proper morphisms (Sam06, Theorem 3.1):

Theorem 3.3.2. *Let $f : X \longrightarrow B$ be a flat proper morphism and $\{\mathcal{K}_1, \ldots, \mathcal{K}_N\} \subset D^b(X)$ objects such that for every $b \in B$ their restrictions*

$$\{\mathcal{K}_1|_{f^{-1}(b)}, \ldots, \mathcal{K}_N|_{f^{-1}(b)}\} \subset D^b(f^{-1}(b))$$

are a full exceptional collection for $D^b(f^{-1}(b))$. Then there exist fully faithful embeddings

$$\begin{aligned} \phi_i : D^b(B) &\hookrightarrow D^b X \\ \mathcal{E} &\longmapsto f^*\mathcal{E} \otimes \mathcal{K}_i \end{aligned} \tag{3.3.3}$$

and the following semiorthogonal decomposition of $D^b(X)$:

$$D^b(X) = \langle f^* D^b(B) \otimes \mathcal{K}_1, \dots f^* D^b(B) \otimes \mathcal{K}_N \rangle. \tag{3.3.4}$$

In this context, the following notion is often used:

Definition 3.3.3. *Let $\pi : X \longrightarrow B$ be a morphism of smooth projective varieties. We say $\mathcal{E} \in D^b(X)$ is* relatively exceptional *over B if the following holds:*

$$\pi_* R\mathcal{H}om(\mathcal{E}, \mathcal{E}) \simeq \mathcal{O}_B. \tag{3.3.5}$$

3.3.2 Orlov's blowup formula

Let us consider a smooth projective variety X with a smooth subvariety $Y \subset X$ of codimension r, and a blowup $\beta : \widehat{X} \longrightarrow X$ with center Y and exceptional locus E. The normal bundle $\mathcal{N}_{Y|X}$ is a rank r vector bundle and $E = \mathbb{P}(\mathcal{N}_{Y|X})$. One has the following diagram:

$$\begin{tikzcd} E \arrow[r, hook, "j"] \arrow[d, "p"] & \widehat{X} \arrow[d, "\beta"] \\ Y \arrow[r, hook, "i"] & X \end{tikzcd} \tag{3.3.6}$$

Let us now call L the pushforward to $\widehat{X}$ of the Grothendieck line bundle on E defined by the projectivization $E = \mathbb{P}(\mathcal{N}_{Y|X})$. We have the following result:

Theorem 3.3.4. *(Orl92, Assertion 4.2, Theorem 4.3) In the notation of Diagram 3.3.6, the following hold:*

- *the functor b is fully faithful*
- *the functor*

$$\begin{aligned} D^b(Y) &\longrightarrow D^b(\widehat{X}) \\ \mathcal{E} &\longmapsto j_*(p^*\mathcal{E} \otimes L^{\otimes(-k)}) \end{aligned} \tag{3.3.7}$$

  is fully faithful for $1 \leq k \leq r-1$

- *one has the following Lefschetz semiorthogonal decomposition:*

$$D^b(\widehat{X}) = \langle j_*(p^*D^b(Y) \otimes L^{\otimes(-r+1)}), \ldots, j_*(p^*D^b(Y) \otimes L^{\otimes(-1)}), \beta^*D^b(X)\rangle \tag{3.3.8}$$

3.3.3 Cayley trick

As above, let us consider a vector bundle $\pi : \mathcal{E} \longrightarrow B$ of rank r and its projectivization $X = \mathbb{P}(\mathcal{E})$, call L the associated Grothendieck line bundle. Fix a regular section $\sigma \in H^0(X, L)$ and call $M = Z(\sigma) \subset X$ and $Y = Z(\pi_*\sigma) \subset B$. We get the following diagram:

$$\begin{tikzcd} \widetilde{Y} \arrow[r, hook, "j"] \arrow[d, "p"] & M \arrow[r, hook, "k"] \arrow[d, "q"] & X \arrow[dl, "\pi"] \\ Y \arrow[r, hook, "i"] & B & \end{tikzcd} \tag{3.3.9}$$

where $q = \pi|_M$ and $p = q|_{q^{-1}(Y)}$. Note that q has the following fibers:

$$q^{-1}(b) \simeq \begin{cases} \mathbb{P}^{r-1} & b \in Y \\ \mathbb{P}^{r-2} & b \in B \setminus Y \end{cases} \tag{3.3.10}$$

and its restriction to the preimage of Y is the projectivization of the normal bundle $\mathcal{N}_{Y|B} \simeq \mathcal{E}|_Y$, hence $\widetilde{Y} \simeq \mathbb{P}(\mathcal{E}|_Y)$.

Theorem 3.3.5. *(Orl03, Proposition 2.10) Let the notation be the one of Diagram 3.3.9. Then there exists the following Lefschetz semiorthogonal decomposition for $D^b(M)$:*

$$D^b(M) = \langle j_* p^* D^b(Y), q^* D^b(B) \otimes L, \ldots, q^* D^b(B) \otimes L^{\otimes(r-1)} \rangle. \tag{3.3.11}$$

3.4 Homological projective duality and related constructions

Introduced by Kuznetsov in (Kuz07), Homological Projetive Duality (HPD) is one of the most useful tools for the manipulation of derived categories of coherent sheaves and semiorthogonal decpomposition. Given a smooth variety X and a morphism $f : X \longrightarrow \mathbb{P}(V)$ to a projective space, the main idea is to construct the derived category of the universal hyperplane section $\mathcal{H}_{\mathbb{P}(V),f}$ associated to f, and determine a subategory $C \subset D^b(\mathcal{H}_{\mathbb{P}(V),f})$ compatible with a given semiorthogonal decomposition of $D^b(X \times \mathbb{P}(V))$. If C is *geometrical*, i.e. if there exists a variety Y such that $D^b(Y) \simeq C$, we say that Y is the homologically projective dual to X. Moreover, linear sections of X and Y by mutually orthogonal hyperplanes share interesting properties. The scope of homological projecitve duality has been pushed much further by introducing the notion of *categorical joins*, allowing to prove derived equivalences of intersection of varieties of the type described in (OR17; BCP20; Man17). In this section we will give a brief survey of HPD.

3.4.1 Universal hyperplane sections and HPD

Let us first define the notion of universal hyperplane section, which will be useful also for different purposes in Chapter 10.

Definition 3.4.1. *Let X be a smooth variety and $V^\vee \subset H^0(X, \mathcal{O}_X(1))$ a vector space of dimension $n+1$. Fix a morphism $f : X \longrightarrow \mathbb{P}(V)$. We call* universal hyperplane section *of X with respect to f the fiber product $\mathcal{H}_{X,f} := X \times_{\mathbb{P}(V)} F(1, n, V)$.*

Remark 3.4.2. The variety $F(1, n, V)$ is often called *incidence quadric* of $\mathbb{P}(V) \times \mathbb{P}(V^\vee)$. In fact, one has $F(1, n, V) = \{(x, y) \in \mathbb{P}(V) \times \mathbb{P}(V^\vee) : y(x) = 0\}$ which characterizes such flag variety as a quadric hypersurface in $\mathbb{P}(V) \times \mathbb{P}(V^\vee)$. Note that for any vector space V, the flag variety $F(1, n, V)$ is the universal hyperplane section of $\mathbb{P}(V)$.

Let us momentarily specialize Definition 3.4.1 to the case where $V^\vee = H^0(X, \mathcal{O}_X(1))$, so that f is a proper embedding. In that case, one gets a more geometric characterization of the universal hyperplane section:

$$\mathcal{H}_X = \{(x, \sigma) \in X \times \mathbb{P}(H^0(X, \mathcal{O}_X(1))) | \sigma(x) = 0\}. \tag{3.4.1}$$

Now, suppose that X has a Lefschetz semiorthogonal decomposition with respect to an ample line bundle $\mathcal{O}_X(1)$ which satisfies $\mathcal{O}_X(1) = f^*\mathcal{O}_{\mathbb{P}(V)}(1)$:

$$D^b(X) = \langle \mathcal{A}_0, \mathcal{A}_1(1), \ldots, \mathcal{A}_{m-1}(m-1) \rangle \tag{3.4.2}$$

By a simple application of Theorem 3.3.1, one has the following semiorthogonal decomposition:

$$D^b(X \times \mathbb{P}(V^\vee)) = \langle \mathcal{A}_0(0) \boxtimes D^b(\mathbb{P}(V^\vee)), \ldots, \mathcal{A}_{m-1}(m-1) \boxtimes D^b(\mathbb{P}(V^\vee)) \rangle. \tag{3.4.3}$$

This semiorthogonal decomposition is $D^b(\mathbb{P}(V^\vee))$-linear by construction.

Definition 3.4.3. *Let X be a smooth variety and $V^\vee \subset H^0(X, \mathcal{O}_X(1))$ a vector space of dimension n. Fix a morphism $f : X \longrightarrow \mathbb{P}(V)$ and suppose there exists a Lefschetz decomposition as in Equation 3.4.2. Supppose $n > m$. Then we call* homological projective dual category *to $D^b(X)$ the category $\mathcal{C}$ appearing in the following semiorthogonal decomposition:*

$$D^b(\mathcal{H}_{X,f}) = \langle \mathcal{C}, \mathcal{A}_1(1) \boxtimes D^b(\mathbb{P}(V^\vee)), \ldots, \mathcal{A}_{m-1}(m-1) \boxtimes D^b(\mathbb{P}(V^\vee)) \rangle. \tag{3.4.4}$$

The existence of $\mathcal{C}$ is a simple matter of definition, as it appears in Equation 3.4.4 as semiorthogonal complement of the blocks of the form $\mathcal{A}_i(i) \boxtimes D^b(\mathbb{P}(V^\vee))$. However, the existence of the semiorthogonal decomposition 3.4.4 is a nontrivial statement (Kuz07, Lemma 5.3). While $\mathcal{C}$ can be an interesting object by its own right, we are mainly interested in the situation when $\mathcal{C}$ is *geometrical*, i.e. it is the derived category of coherent sheaves of a variety:

Definition 3.4.4. *Let $f : X \longrightarrow \mathbb{P}(V)$ be a morphism defined on a smooth variety X to a projective space $\mathbb{P}(V)$ as above and let $\mathcal{C}$ be the HPD category of X with respect to f. Let Y be a variety with a morphism $f : Y \longrightarrow \mathbb{P}(V^\vee)$. We say that Y is* homologically projective dual *to X if there exists an equivalence of categories $\phi : D^b(Y) \longrightarrow \mathcal{C}$ which is $\mathbb{P}(V^\vee)$-linear.*

As we mentioned above, homological projective duality behaves well under the action of taking linear sections. Let us consider an *admissible* linear subspace $L \subset V^\vee$, i.e. a subspace such that $X_L := X \times_{\mathbb{P}(V)} \mathbb{P}(L^\perp)$ and $Y_L := Y \times_{\mathbb{P}(V^\vee)} \mathbb{P}(L^\perp)$ have dimension respectively $\dim(X) - \dim(L)$ and $\dim(Y) + \dim(L) - n - 1$. Then, one has the following theorem:

Theorem 3.4.5. *(Kuz07, Theorem 6.3) Let $f : X \longrightarrow \mathbb{P}(V)$ and $g : Y \longrightarrow \mathbb{P}(V^\vee)$ be as above, with Y homologically projective dual to X. Let X have a semirothogonal decomposition as in Equation 3.4.2 where we call $\mathcal{A}_k = \langle \mathfrak{a}_k, \dots, \mathfrak{a}_{m-1} \rangle$ for $0 \leq k \leq m-1$, once defined $\mathfrak{a}_k$ to be the right orthogonal of $\mathcal{A}_{k+1}$ in $\mathcal{A}_k$. Then the following statements are true:*

1. *Y is smooth and it admits the following semiorthogonal decomposition for some positive integer l:*

$$D^b(Y) = \langle \mathcal{B}_{l-1}(1-l), \dots, \mathcal{B}_1(-1), \mathcal{B}_0 \rangle \tag{3.4.5}$$

   where we defined the block $\mathcal{B}_k = \langle \mathfrak{a}_0, \dots, \mathfrak{a}_{n-k-2} \rangle$.

2. *For every admissible linear subspace $L \subset V^\vee$ of dimension r one*

has the following semiorthogonal decompositions:

$$\begin{aligned} D^b(X_L) &= \langle \mathcal{C}_L, \mathcal{A}_r(1), \dots, \mathcal{A}_{m-1}(m-r) \rangle \\ D^b(Y_L) &= \langle \mathcal{B}_{l-1}(n+1-r-l), \dots, \mathcal{B}_{n-r}(-1), \mathcal{C}_L \rangle \end{aligned} \tag{3.4.6}$$

3.4.2 HPD for projective bundles

An interesting setting where to apply Theorem 3.4.5 can be given by choosing $X = \mathbb{P}(\mathcal{E})$ where $\mathcal{E}$ is a vector bundle over a smooth projective variety B. Let $V^\vee \subset H^0(B, \mathcal{E})$ be a space of sections generating $\mathcal{E}$, so that we have a surjective morphism of vector bundles on B:

$$V^\vee \otimes \mathcal{O} \longrightarrow \mathcal{E} \longrightarrow 0. \tag{3.4.7}$$

Hence, we can construct a morphism of projective varieties:

$$f : X \longrightarrow \mathbb{P}(V). \tag{3.4.8}$$

Then, one can find a semiorthogonal decomposition for X by Theorem 3.3.1, and this gives rise to a (naturally $\mathbb{P}(V)$-linear) semiorthogonal decompposition for $X \times \mathbb{P}(V)$. It turns out (Kuz07, Section 8) that a nice semiorthogonal decomposition can also be found for $\mathcal{H}_{X,f}$, which proves that the homological projective dual category of X is always geometric and it is the derived category of another projective bundle.

Lemma 3.4.6. *(Kuz07, Lemma 8.1) Let $\mathcal{E} \longrightarrow B$ be a vector bundle of rank r over a smooth projective variety of dimension n and let $V^\vee \subset H^0(B, \mathcal{E})$ be an N-dimensional vector space of sections which generates $\mathcal{E}$. Call $X = \mathbb{P}(\mathcal{E})$ and $f : X \longrightarrow \mathbb{P}(V)$ as above. Then the homological* *projective dual variety of X has codimension r inside* $\mathbb{P}(V) \times B$ *and it is given by* $Y = \mathbb{P}(\mathcal{E}^{\perp})$*, where we call:*

$$\mathcal{E}^{\perp} = \ker(V \otimes \mathcal{O} \longrightarrow \mathcal{E})^{\vee}. \tag{3.4.9}$$

An accurate choice of rank and dimension of the space of sections can provide examples of homologically projective self-dual projective bundles.

Proposition 3.4.7. *The projective bundle* $\mathbb{P}(\wedge^3 T(-3))$ *over* $\mathbb{P}^5$ *is homologically projective self-dual.*

Proof. Let us fix $\mathbb{P}^5 = \mathbb{P}(V_6)$. The vector bundle $\wedge^3 T(-3)$ has rank 10, it is globally generated and it has a space of global sections $H^0(\mathbb{P}^5, \wedge^3 T(-3)) = \wedge^3 V_6^{\vee}$. This comes from the fact that the dual Euler sequence reads:

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow V_6^{\vee} \otimes \mathcal{O} \longrightarrow T(-1) \longrightarrow 0 \tag{3.4.10}$$

and its third symmetric power gives:

$$\begin{aligned} 0 \longrightarrow \mathcal{O}(-3) \longrightarrow V_6^{\vee} \otimes \mathcal{O}(-2) \longrightarrow \wedge^2 V_6^{\vee} \otimes \mathcal{O}(-1) \longrightarrow \\ \longrightarrow \wedge^3 V_6^{\vee} \otimes \mathcal{O} \longrightarrow \wedge^3 T(-3) \longrightarrow 0 \end{aligned} \tag{3.4.11}$$

from which we get the surjection

$$\wedge^3\, V_6^{\vee} \otimes \mathcal{O} \xrightarrow{g} \wedge^3 T(-3) \longrightarrow 0 \tag{3.4.12}$$

with kernel $\wedge^2 T(-2)$.

Hence, g gives rise to a morphism $f : \mathbb{P}(\wedge^3 T(-3)) \longrightarrow \mathbb{P}(\wedge^3 V_6^{\vee})$. In light of Equation 3.4.12, an application of Lemma 3.4.6 yields that the homological projective dual of $\mathbb{P}(\wedge^3 T(-3))$ is $\wedge^2 T^{\vee}(2)$, but one has $\wedge^2 T^{\vee}(2) \simeq \wedge^3 T(2) \otimes \det(T^{\vee}) \simeq \wedge^3 T(-3)$. $\square$

3.5 Derived equivalence and birational equivalence: DK conjectures

Let us consider a birational map $\mu : X_1 \dashrightarrow X_2$ resolved by two morphisms $f_i : X_0 \longrightarrow X_i$, where X_0 is smooth projective. We get the following diagram:

$$\begin{tikzcd} & X_0 \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\ X_1 \arrow[rr, dashed, "\mu"] & & X_2 \end{tikzcd} \tag{3.5.1}$$

We are interested in the following two cases:

- If $f_1^*\omega_{X_1} \simeq f_2^*\omega_{X_2}$, we say that μ is a K-*equivalence*. We say that a K-equivalence is *simple* if f_1 and f_2 are blowups.
- If $f_1^*K_{X_1} + E \sim f_2^*K_{X_2}$ or $f_1^*K_{X_1} \sim f_2^*K_{X_2} + E$ for some effective divisor E, we say that μ is a K-*inequality*.

Observe that any birational map μ fitting in a diagram like 3.5.1 such that X_1 and X_2 have trivial canonical bundle is a K-equivalence.
A natural question, justified by a multitude of positive examples and by the lack of counterexamples, is to ask whether K-equivalence implies D-equivalence, and whether a K-inequality implies an embedding of categories. In fact, this is known as the DK conjecture, which we state here below:

Conjecture 3.5.1. *(BO02; Kaw02) The following statements are true:*

- *Let $\mu : X_1 \dashrightarrow X_2$ be a K-inequality such that $f_1^*\omega_{X_1} \otimes f_2^*\omega_{X_2}^\vee$ is ample. Then, there exists a fully faithful functor $D^b(X_2) \longrightarrow D^b(X_1)$.*

- *Let $\mu : X_1 \dashrightarrow X_2$ be a K-equivalence. Then $D^b(X_1) \simeq D^b(X_2)$.*

Conjecture 3.5.1 is known to be true in several cases, among them we list the following:

- Standard flips and flops (BO02): here f_1 and f_2 are smooth blowups, respectively in $\mathbb{P}^k$ and $\mathbb{P}^l$, for some $k, l \in \mathbb{N}$. The exceptional locus for both the blowups is $E \simeq \mathbb{P}^k \times \mathbb{P}^l$. If $k = l$ this is called a flop, and X_1 and X_2 are derived equivalent. On the other hand, if $l < k$ it is called a flip, and one has $D^b(X_2) \subset D^b(X_1)$.

- Mukai flops: Also here f_1 and f_2 are blowups with center isomorphic to projective spaces of the same dimension, but their exceptional loci are isomorphic to the partial flag variety $F(1, n, n+1)$. Derived equivalence has been proved by Namikawa (Nam03)

- K-equivalent threefolds are derived equivalent (Bri02, Theorem 1.1)

We will provide new examples supporting Conjecture 3.5.1 in Chapter 11, alongside with new proofs for some of the known examples.

4 Homogeneous roof bundles

The purpose of the present chapter is to introduce the main geometric setting for the rest of this work. Motivated by the study of DK-conjectures (see Section 3.5), Kanemitsu introduced the notion of roof, or roof of projective bundles, which lies at the core of the classification of simply K-equivalent maps. Here we will review such notion with particular emphasis to homogeneous roofs and locally trivial families of these objects over smooth projective varieties.

4.1 Homogeneous roofs

Let us begin by recalling the definition of the following special class of Mukai pairs (Muk88):

Definition 4.1.1. (Kan18, Definition 0.1) *A* simple Mukai pair $(Z_1, \mathcal{E})$ *is the data of a Fano variety Z_1 of Picard number one and an ample vector bundle $\mathcal{E}$ over Z_1 such that:*

- $\det(\mathcal{E}) \simeq \omega_Y^\vee$
- $\mathbb{P}(\mathcal{E})$ *admits another $\mathbb{P}^{r-1}$-bundle structure, where* $r = \mathrm{rk}(\mathcal{E})$.

Definition 4.1.2. (Kan18, Definition 0.1) *A* roof *of rank r, or* roof of $\mathbb{P}^{r-1}$-bundles*, is a Fano variety X which is isomorphic to the projectivization of a rank r vector bundle $\mathcal{E}$ over a Fano variety Z_1, where $(Z_1, \mathcal{E})$ is a simple Mukai pair.*

Remark 4.1.3. An equivalent definition for a roof of projective bundles is a Fano variety of Picard rank 2 and index r equipped with two different $\mathbb{P}^{r-1}$-fibration structures (Kan18, page 2).

The following proposition provides a useful characterization of roofs:

Proposition 4.1.4. *(Kan18, Proposition 1.5) Let X be a smooth projective Fano variety of Picard number two. Assume that for $i \in \{1;2\}$ the extremal contraction $X \longrightarrow Z_i$ is a smooth $\mathbb{P}^{r-1}$-fibration. Then the following are equivalent:*

- *X is a roof*
- *The index of X is r*
- *There exists a line bundle L on X which restricts to $\mathcal{O}(1)$ on every fiber of each extremal contraction.*

Given a roof X, the following picture emerges:

$$\begin{tikzcd} & \mathbb{P}(\mathcal{E}) = X = \mathbb{P}(\mathcal{F}) \arrow[dl, "h_1"'] \arrow[dr, "h_2"] & \\ Z_1 & & Z_2 \end{tikzcd} \tag{4.1.1}$$

Among roofs, nearly all known examples can be described in terms of G-homogeneous varieties of Picard number two where G is a semisimple Lie group, with the projective bundle structures defined by the natural surjections to G-Grassmannians:

Definition 4.1.5. *A* homogeneous roof *is a roof which is isomorphic to a homogeneous variety G/P of Picard number two, where G is a semisimple Lie group and P is a parabolic subgroup.*

The data of a homogeneous roof defines the following diagram:

$$\begin{tikzcd} & G/P \arrow[dl, "h_1"'] \arrow[dr, "h_2"] & \\ G/P_1 & & G/P_2 \end{tikzcd} \tag{4.1.2}$$

This class of homogeneous roofs has remarkable properties: for example, as we will clarify below, a general hyperplane section of a homogeneous roof defines a pair of Calabi–Yau varieties, which are conjectured to be derived equivalent (KR20, Conjecture 2.6). In the present work we will mostly focus on homogeneous roofs.

A complete list of homogeneous roofs has been given in (Kan18, Section 5.2.1). Let us summarize its content in Table 4.1. In the column "type" we refer to the nomenclature introduced by Kanemitsu, which will also be adopted throughout the reminder of this work. Hereafter, given a semisimple Lie group G, $G/P^{n_1,\dots,n_k}$ denotes the quotient of G by its parabolic subgroup such that the Levi factor of the corresponding Lie algebra is the union of root spaces related to the simple roots $n_1, \dots n_k$. The expressions G/P_1 and G/P_2 will denote the images of the two $\mathbb{P}^{r-1}$-bundle structures h_1 and h_2 of the roof G/P. Where it is possible, we use the more standard notations for (orthogonal and symplectic) Grassmannians and flag varieties.

G	type	G/P	G/P_1	G/P_2
$SL(k+1)\times SL(k+1)$	$A_k \times A_k$	$\mathbb{P}^k \times \mathbb{P}^k$	$\mathbb{P}^k$	$\mathbb{P}^k$
$SL(k+1)$	A_k^M	$F(1,k,k+1)$	$\mathbb{P}^k$	$\mathbb{P}^k$
$SL(2k+1)$	A_{2k}^G	$F(k,k+1,2k+1)$	$G(k,2k+1)$	$G(k+1,2k+1)$
$Sp(3k-2)$ (k even)	$C_{3k/2-1}$	$IF(k-1,k,3k-2)$	$IG(k-1,3k-2)$	$IG(k,3k-2)$
$Spin(2k)$	D_k	$OG(k-1,2k)$	$OG(k,2k)^+$	$OG(k,2k)^-$
F_4	F_4	$F_4/P^{2,3}$	F_4/P^2	F_4/P^3
G_2	G_2	$G_2/P^{1,2}$	G_2/P^1	G_2/P^2

Table 4.1: Homogeneous roofs

Lemma 4.1.6. *Let X be a homogeneous roof of $\mathbb{P}^{r-1}$-bundles with structure morphisms $h_i : X \longrightarrow Z_i$ and consider a general section $\sigma \in H^0(X, L)$, where L is the line bundle of Proposition 4.1.4. Call $\mathcal{E}_i := h_{i*}L$. Then $Y_i = Z(h_{i*}\sigma) \subset Z_i$ is either empty or a Calabi–Yau variety of codimension r (in the sense of Remark 2.5.2).*

Proof. The proof follows from Lemma 2.5.3. and generic smoothness of σ. □

Remark 4.1.7. Observe that in Lemma 4.1.6 the case $Y_i = \emptyset$ is represented only by roofs of type $A_k \times A_k$. In fact, for these roofs, the projective bundle structures are given by projectivizations of vector bundles of rank $k+1$ on $\mathbb{P}^k$, hence the zero loci of pushforwards of a general section $\sigma \in H^0(\mathbb{P}^k \times \mathbb{P}^k, L)$ are empty. In all other cases, the zero loci have nonnegative expected dimension.

Definition 4.1.8. (cfr. (KR20, Definition 2.5)) *Let X be a homogeneous roof of type different from $A_k \times A_k$, fix the notation of Lemma 2.5.3. We say (Y_1, Y_2) is a Calabi–Yau pair associated to the roof X if $Y_1 \simeq Z(h_{1*}\sigma)$ and $Y_2 \simeq Z(h_{2*}\sigma)$ where $\sigma \in H^0(X, L)$ is a general section.*

Example 4.1.9. Let V_n be an n-dimensional vector space. We denote by $F(1, n-1, V_n)$ the flag variety parametrizing pairs of a subspace of dimension 1 contained in a subspace of dimension $n-1$ of V_n. Then, $F(1, n-1, V_n)$ has natural projections to the Grassmannians $G(1, V_n)$ and $G(n-1, V_n)$, which are both isomorphic to $\mathbb{P}^{n-1}$. This picture is summarized by the following diagram:

$$\begin{tikzcd} & F(1,n-1,V_n) \arrow[dl, "h_1"'] \arrow[dr, "h_2"] & \\ G(1,V_n) & & G(n-1,V_n) \end{tikzcd} \tag{4.1.3}$$

Then one has:

- $\mathbb{P}(h_{1*}L) \simeq \mathbb{P}(h_{2*}L) \simeq F(1, n-1, V_n)$
- $\det(h_{1*}L) = \mathcal{O}(n) = \omega^{\vee}_{G(1,V_n)}$

hence, $F(1, n-1, V_n)$ is a roof $\mathbb{P}^{n-1}$-bundles. It appears in the list of (Kan18) as roof of type A^M_{n-1}.

4.2 Non homogeneous roof

In the following section we will describe the only non homogeneous roof present in the list of (Kan18) , and we will describe the $K3$ pairs arising by such construction.

4.2.1 Homogeneous vector bundles on the five dimensional quadric

Let $Q \subset \mathbb{P}^6$ be a quadric hypersurface of dimension five, let $\mathcal{S}$ be its rank 4 spinor bundle. Ottaviani constructed a 7-dimensional moduli space of rank 3 bundles $\mathcal{G}$ such that

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{S}^\vee \longrightarrow \mathcal{G} \longrightarrow 0 \tag{4.2.1}$$

More precisely, there exists a moduli space isomorphic to $\mathbb{P}^7 \setminus Q_6$ of rank 3 vector bundles $\mathcal{G}$ with Chern class $c(\mathcal{G}) = (2, 2, 2)$, and those bundles are the ones satisfying Equation 4.2.1 (Ott88, Theorem 3.2).
By the Borel–Weil–Bott theorem and the sequence 4.2.1, one proves that $\dim H^0(Q, \mathcal{G}(1)) = 41$ and by the above we have $c(\mathcal{G}(1)) = (5, 9, 12)$. Hence, a section s in such 41-dimensional vector space defines a $K3$ surface of degree 12 in $\mathbb{P}^6$.

Let $\mathbb{O}$ denote the complexified Cayley octonions (for details see (Kan19, Definition 2.4) and the source therein). It is known by (Kan19, Theorem 2.6) that the projectivization of the Ottaviani bundle can be described in the following way:

$$\mathbb{P}(\mathcal{G}^\vee) = \{(x, y) \in \mathbb{P}(\operatorname{Im}\mathbb{O}^\vee) \times \mathbb{P}(\operatorname{Im}\mathbb{O}^\vee) | x \cdot x = y \cdot y = x \cdot y = 0\} := X \tag{4.2.2}$$

This variety has two natural projections to the quadrics $Q = \{x \in \mathbb{P}(\operatorname{Im}\mathbb{O}^\vee) | x \cdot x = 0\}$ and $\widetilde{Q} = \{y \in \mathbb{P}(\operatorname{Im}\mathbb{O}^\vee) | y \cdot y = 0\}$ leading to the following diagram:

$$\begin{tikzcd} & X \arrow[dl, "\pi"'] \arrow[dr, "\widetilde{\pi}"] & & \\ Y \arrow[r, hook] & Q & \widetilde{Q} & \widetilde{Y} \arrow[l, hook'] \end{tikzcd} \tag{4.2.3}$$

where both Y and $\widetilde{Y}$ are $K3$ surfaces described as zero loci of (twisted) Ottaviani bundles $\mathcal{G}(1)$ and $\widetilde{\mathcal{G}}(1)$, and $\mathbb{P}(\mathcal{G}^\vee(-1)) \simeq \mathbb{P}(\widetilde{\mathcal{G}}^\vee(-1)) \simeq X$.

Remark 4.2.1. Diagram 4.2.3 appears as the roof of type $G_2^\dagger$ in the list of (Kan18, Section 5.2.1), and it is the only non homogeneous example of such construction. In fact,the Fano 7-fold X, in contrast with the other examples, is not a generalized flag. However, if we consider the surjection from the G_2 flag to the five dimensional quadric, we obtain the projectivization of a rank two vector bundle, which admits a second projective bundle structure alongside with a surjection to the G_2-Grassmannian. This construction yields the roof of type G_2 studied in (IMOU19; Kuz18), which gives derived equivalent but non isomorphic Calabi–Yau threefolds.

4.2.2 The roof of type D_4 and its degeneration

Recall the homogeneous roof of type D_4.

$$\begin{tikzcd} & X_{D_4} \arrow[dl, "\pi"'] \arrow[dr, "\widetilde{\pi}"] & & \\ Y \arrow[r, hook] & Q_6 & \widetilde{Q}_6 & \widetilde{Y} \arrow[l, hook'] \end{tikzcd} \tag{4.2.4}$$

Here Q_6 and $\widetilde{Q}_6$ are six-dimensional quadrics representing spinor varieties $OG(4,8)_\pm$ and $X_{D_4} = OG(3,8) = \mathbb{P}_{Q_6}(\mathcal{S}^\vee(1)) = \mathbb{P}_{\widetilde{Q}_6}(\widetilde{\mathcal{S}}^\vee(1))$ where $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ are the spinor bundles respectively on Q_6 and $\widetilde{Q}_6$. Note that $OG(3,8)$ admits two different projective bundle structures given by the maps π and $\widetilde{\pi}$, which can be interpreted as the projections determined by the embeddings of the parabolic subgroup of $OG(3,8)$ inside the parabolic subgroups defining $OG(4,8)_\pm$.
There exists the following short exact sequence on Q_6 (Ott88, Section 3), whose restriction to Q_5 is Equation 4.2.1:

$$0 \longrightarrow \mathcal{O}(1) \longrightarrow \mathcal{S}^\vee(1) \longrightarrow \mathcal{G}(1) \longrightarrow 0. \tag{4.2.5}$$

Note that when the Mukai pair moves in a moduli we get a family of roofs. In this way one can also obtain degenerations of roofs which involve bundles which are not necessarily stable. This is the case for instance in the following context.

Degeneration of roofs

Considering the family of extensions between $\mathcal{O}(1)$ and $\mathcal{G}(1)$ we see the trivial extension $\mathcal{O}(1) \oplus \mathcal{G}(1)$ as a degeneration of $\mathcal{S}^\vee(1)$. It follows that X_{D_4} admits a degeneration to $\hat{X}_{D_4} = \mathbb{P}_{Q_6}(\mathcal{O}(1) \oplus \mathcal{G}(1))$. The latter variety is not a roof, but it admits a natural surjection to Q_6. A general hyperplane section of $\hat{X}_{D_4}$ now gives rise to a $K3$ surface obtained as a zero locus of a section of $\mathcal{O}(1) \oplus \mathcal{G}(1)$ on such quadric. In consequence the $K3$ is given as the zero locus of the restriction of the corresponding section of $\mathcal{G}(1)$ to a five-dimensional quadric Q_5 obtained as a hyperplane section of Q_6. If we now consider the restriction of $\mathcal{G}(1)$ to the zero locus of a section of $\mathcal{O}(1)$ we obtain the roof $G_2^\dagger$. The latter roof is a subvariety of some degeneration of the roof of type D_4. Moreover, the $K3$ surfaces associated to this roof are degenerations of $K3$ surfaces associated to X_{D_4}. We however see in Subsection 4.2.3 that a general $K3$ surface of degree 12 appears also in the degenerate description.

Remark 4.2.2. Note that we can further degenerate the $G_2^\dagger$ roof using the exact sequence:

$$0 \longrightarrow \mathcal{C}(2) \longrightarrow \mathcal{G}(1) \longrightarrow \mathcal{O}(2) \longrightarrow 0. \tag{4.2.6}$$

where $\mathcal{C}$ is the Cayley bundle on Q_5. The zero locus of $\mathcal{C}(2) \oplus \mathcal{O}(2)$ is the intersection of a del Pezzo threefold of degree 6 with a quadric. We can then consider the restriction of $\mathcal{C}(2)$ to the zero locus of a section of $\mathcal{O}(2)$ which is just a complete intersection of two quadrics. This is however not a roof as it does not appear in the classification of (Kan18, Theorem 5.12). The $K3$ surfaces obtained in this way are also not general $K3$ surfaces of degree 12 as their Picard number is ≥ 2.

4.2.3 Completeness of the family of K3 varieties of type $G_2^\dagger$

In the remainder of this section we prove that the family of $K3$ surfaces described as sections of an Ottaviani bundle $\mathcal{G}(1)$ represents a dense open subset of the family of polarized $K3$ surfaces of degree 12. In particular the general element of this family has Picard number one. We then prove that pairs of $K3$ surfaces associated to the roof $G_2^\dagger$ are in general not isomorphic.

It is well-known (Muk87, Corollary 0.3) that a polarized $K3$ surfaces of degree $2g-2$ has an embedding (defined by its polarization) in the projective space $\mathbb{P}^g$. If we can prove that our degree 12 $K3$ surfaces in $\mathbb{P}^6$ form a 26-dimensional family up to automorphisms of $\mathbb{P}^6$, then our family can be recovered by the complete 19-dimensional family in $\mathbb{P}^7$ by means of a projection from one point. Since the general element of a complete family of $K3$ surfaces has Picard number one, we conclude that the same holds for our family.

Lemma 4.2.3. *Let $Q \subset \mathbb{P}^6$ be a five dimensional quadric hypersurface and $\mathcal{G}$ an Ottaviani bundle on Q. If $Y = Z(s)$ for $s \in H^0(Q, \mathcal{G}(1))$, then Y determines the bundle $\mathcal{G}$ and the section s up to scalar multiplication.*

Proof. Let us consider a $K3$ surface $Y \subset Q$ and let $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ be two Ottaviani bundles on Q, such that there exist two sections $s \in H^0(Q, \mathcal{G}(1))$ and $\widetilde{s} \in H^0(Q, \widetilde{\mathcal{G}}(1))$ with $Y = Z(s) = Z(\widetilde{s})$. Then we have the following diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{G}^\vee(-1) \arrow[r, "\alpha_s"] \arrow[d, "\beta"] & \mathcal{I}_{Y|Q} \arrow[r] \arrow[d, equal] & 0 \\
\cdots \arrow[r] & \widetilde{\mathcal{G}}^\vee(-1) \arrow[r, "\alpha_{\widetilde{s}}"] & \mathcal{I}_{Y|Q} \arrow[r] & 0
\end{tikzcd} \tag{4.2.7}$$

where $\mathcal{I}_{Y|Q}$ is the ideal sheaf of $Y \subset Q$ and the rows are given by the Koszul resolutions of $\mathcal{I}_Y$ with respect to the two sections. The existence of β is a consequence of the following claim.

Claim. *The following map is surjective:*

$$\operatorname{Hom}(\mathcal{G}^\vee(-1), \widetilde{\mathcal{G}}^\vee(-1)) \longrightarrow \operatorname{Hom}(\mathcal{G}^\vee(-1), \mathcal{I}_{Y|Q})$$

Proving the claim is equivalent to show that

$$f : H^0(Q, \mathcal{G} \otimes \mathcal{G}^\vee) \longrightarrow H^0(Q, \mathcal{G}(1) \otimes \mathcal{I}_{Y|Q}) \tag{4.2.8}$$

is surjective. To this purpose, we compute the tensor product of $\mathcal{G}(1)$ with the Koszul resolution of $\mathcal{I}_Y$ with respect to $\widetilde{s}$. Using $\det \widetilde{\mathcal{G}}^\vee = \mathcal{O}(-2)$ and $\wedge^2 \widetilde{\mathcal{G}}^\vee \simeq \widetilde{\mathcal{G}}(-2)$ we find:

$$0 \longrightarrow \mathcal{G}(-4) \longrightarrow \mathcal{G} \otimes \widetilde{\mathcal{G}}(-3) \longrightarrow \mathcal{G} \otimes \widetilde{\mathcal{G}}^\vee \longrightarrow \mathcal{G}(1) \otimes \mathcal{I}_{Y|Q} \longrightarrow 0 \tag{4.2.9}$$

Cohomology can be computed with the Borel–Weil–Bott theorem: in fact, using the sequence 4.2.1, we can resolve all the bundles of 4.2.9 in terms of twists of tensor products of $\mathcal{S}$ and its dual. In particular, the leftmost term of 4.2.9 is acyclic beacuse both $\mathcal{O}(-4)$ and $\mathcal{S}^\vee(-4)$ are, while the cohomology of the second one follows from the diagram:

$$\begin{tikzcd}
\mathcal{O}(-3) \arrow[d, hook] & \mathcal{S}^\vee(-3) \arrow[d, hook] & \\
\mathcal{S}^\vee(-3) \arrow[d, two heads] & \mathcal{S}^\vee \otimes \mathcal{S}^\vee(-3) \arrow[d, two heads] & \\
\mathcal{G}(-3) \arrow[r, hook] & \mathcal{S}^\vee \otimes \mathcal{G}(-3) \arrow[r, two heads] & \mathcal{G} \otimes \widetilde{\mathcal{G}}(-3)
\end{tikzcd} \tag{4.2.10}$$

In the first two rows, the only term which is not acyclic is $\mathcal{S}^\vee \otimes \mathcal{S}^\vee(-3) = \wedge^2 \mathcal{S}^\vee(-3) \oplus \mathrm{Sym}^2 \mathcal{S}^\vee(-3)$: the first summand has no cohomology, while the second one has cohomology $\mathbb{C}[-2]$. Let us call $\mathcal{K}$ the cokernel of $\mathcal{G}(-4) \longrightarrow \mathcal{G} \otimes \widetilde{\mathcal{G}}(-3)$. Then one has $H^\bullet(Q, \mathcal{K}) = \mathbb{C}[-2]$, and by the sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{G} \otimes \widetilde{\mathcal{G}}^\vee \longrightarrow \mathcal{G}(1) \otimes \mathcal{I}_{Y|Q} \longrightarrow 0 \tag{4.2.11}$$

we conclude that f is surjective, thus proving the claim.

Since Ottaviani bundles are stable (Ott88, Theorem 3.2), the map β can be either zero or an isomorphism, so we deduce that s and $\widetilde{s}$ must be sections of isomorphic Ottaviani bundles. Hence, the proof is completed by observing that $\mathrm{Hom}(\mathcal{G}, \mathcal{G}) = \mathbb{C}$. □

Lemma 4.2.4. *Let $Y \subset Q$ be a $K3$ surface satisfying the hypotheses of Lemma 4.2.3. Then Y is contained in a unique quadric in $\mathbb{P}^6$.*

Proof. The proof follows from observing that $h^0(Q, \mathcal{I}_{Y|Q}(2)) = 0$. By the Koszul resolution of $\mathcal{I}_{Y|Q}$ and the relation $\mathcal{G}^\vee \simeq \wedge^2\mathcal{G}(-2)$ we find the following exact sequence:

$$0 \longrightarrow \mathcal{O}(-3) \longrightarrow \mathcal{G}(-2) \longrightarrow \mathcal{G}^\vee(1) \longrightarrow \mathcal{I}_{Y|Q}(2) \longrightarrow 0$$

and the desired result is obtained by an application of the Borel–Weil–Bott theorem. In fact, as in the proof of Lemma 4.2.3, one can resolve the first three bundles in terms of twists of $\mathcal{O}$, $\mathcal{S}$ and its dual. □

Proposition 4.2.5. *The general $K3$ surface described as a zero locus of a section of $\mathcal{G}(1)$, where $\mathcal{G}$ is an Ottaviani bundle, has Picard number one.*

Proof. The space of sections of an Ottaviani bundle has dimension 41, and the moduli space of Ottaviani bundles on Q is 7-dimensional. Since the action of $\mathrm{Aut}\, Q = \mathrm{Spin}(7)$ is transitive on the moduli space of Ottaviani bundles, and a $K3$ surface $Y \subset Q$ determines the section, the (projective) dimension of the family is given by:

$$40 - 21 + 7 = 26$$

where $21 - 7$ is the dimension of the space of automorphisms of Q fixing an Ottaviani bundle. Hence, the family we are describing is a 26 dimensional family (of classes up to automorphisms of $\mathbb{P}^6$) of embedded $K3$ surfaces of degree 12 in $\mathbb{P}^6$. Since each $K3$ of degree 12 has a projective embedding in $\mathbb{P}^7$ a complete family of $K3$ of degree 12 in $\mathbb{P}^6$ can be described by a $19+7 = 26$-parameter space, via projection from a point in $\mathbb{P}^7$. This proves that our family is complete, therefore the general element has Picard number one. □

4.3 Roofs, derived equivalence and non-compact DK-conjecture

It is a natural question to ask whether every roof provides pairs of derived equivalent Calabi–Yau manifolds. Such conjecture, which we state here below, is supported by several worked examples, despite the lack of a general proof.

Conjecture 4.3.1. *Let X be a roof. Then there exists a derived equivalence $D^b(Y) \simeq D^b(\widetilde{Y})$, where Y and $\widetilde{Y}$ are Calabi–Yau $n-r$-folds defined by pushforwards of a smooth hyperplane section $M \subset X$.*

Remark 4.3.2. The DK conjecture (Conjecture 3.5.1) states that if two smooth projective varieties are related by a flop, they are derived equivalent (Kaw02), (BO02). Conjecture 4.3.1 is particularly interesting if we observe that the total spaces $\mathcal{E}^\vee$ and $\widetilde{\mathcal{E}}^\vee$ are related by a flop. A positive answer to such conjecture has been given for the roofs of type G_2 by (Ued19), and for the roofs of type C_2 and A_4^G by (Mor19), but again a general proof of the validity of the DK conjecture for bundles related by a roof is missing.

The problem of finding a derived equivalence for the total spaces is strictly related to proving that the Calabi–Yau zero loci are derived equivalent: in fact, one has the following diagram

$$\begin{tikzcd}
 & L^\vee \arrow[dl, "f"'] \arrow[dr, "g"] & \\
\mathcal{E}^\vee & X \arrow[u, hook] \arrow[dl] \arrow[dr] & \widetilde{\mathcal{E}}^\vee \\
Y \subset B \arrow[u, hook] & & \widetilde{B} \supset \widetilde{Y} \arrow[u, hook]
\end{tikzcd} \tag{4.3.1}$$

where f and g are blowups of respectively $\mathcal{E}^\vee$ in B and $\widetilde{\mathcal{E}}^\vee$ in $\widetilde{B}$, the bases are embedded in the total spaces as zero sections. Then it is possible to write the derived category of the total space of $L^\vee$ in two ways, each of them being a semiorthogonal decomposition containing a twist of $D^b(\mathcal{E}^\vee)$ and a twist of $D^b(B)$, or a totally similar decomposition on the other side of the diagram. As we will see, this picture is very similar to Diagram 9.1.1: in fact, in the existing worked examples, the strategy of the proof adopted for the total spaces is the same that has been used for the zero loci (see for example the relation of (Mor19) with (KR17) and of (Ued19) with (Kuz18)).

4.4 Homogeneous roof bundles

While the problem of describing and classifying fibrations of roofs over a smooth projective variety has been addressed in (Kan18; ORS20), we focus on a special class of such objects, which we call *homogeneous roof bundles*: they provide a natural relativization of homogeneous roofs, while keeping many of the properties of the latter objects in a relative setting.

Definition 4.4.1. *Let G be a semisimple Lie group and P a parabolic subgroup such that G/P is a homogeneous roof. Let $\mathcal{V}$ be a principal G-bundle over a smooth projective variety B. We define a* homogeneous roof bundle *over B the variety $\mathcal{V} \times^G G/P$.*

We get this diagram:

$$\begin{tikzcd} & \mathcal{V} \times^G G/P \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathcal{V} \times^G G/P_1 & & \mathcal{V} \times^G G/P_2 \end{tikzcd} \tag{4.4.1}$$

Remark 4.4.2. Note that $\mathcal{V} \times^G G/P$ is a locally trivial fibration over B with fiber $F_b \simeq G/P$.

Lemma 4.4.3. *Let G be a semisimple Lie group and $P \subset G$ a parabolic subgroup such that G/P is a homogeneous roof with projective bundle structures $h_i : G/P \simeq \mathbb{P}(E_i) \longrightarrow G/P_i$ for $i \in \{1; 2\}$. Let $\mathcal{V} \longrightarrow B$ be a principal G-bundle over a smooth projective variety B. Then there are vector bundles $\mathcal{E}_i$ such that the homogeneous roof bundle $\mathcal{Z} = \mathcal{V} \times^G G/P$*

admits projective bundle structures $p_i : \mathcal{Z} \simeq \mathbb{P}(\mathcal{E}_i) \longrightarrow \mathcal{Z}_i$ *for* $i \in \{1; 2\}$ *such that the following diagram is commutative:*

$$\begin{tikzcd} & \mathcal{Z} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathcal{Z}_1 \arrow[dr, "r_1"'] & & \mathcal{Z}_2 \arrow[dl, "r_2"] \\ & B & \end{tikzcd} \tag{4.4.2}$$

where r_1 *and* r_2 *are smooth extremal contractions and* $\mathcal{Z}_i := \mathcal{V} \times^G G/P_i$. *Moreover, there exists a line bundle* $\mathcal{L}$ *on* $\mathcal{Z}$ *such that* $\mathcal{L}$ *restricts to* $\mathcal{O}(1)$ *on the fibers of both* p_1 *and* p_2*, and such that* $p_{i*}\mathcal{L} \simeq \mathcal{E}_i$.

Proof. Let us call $\pi : \mathcal{Z} \longrightarrow B$ the map induced by the structure map $\mathcal{V} \longrightarrow B$. Then, for every $b \in B$ we have $\pi^{-1}(b) \simeq G/P$. We obtain the following diagram:

$$\begin{tikzcd} & \mathcal{Z} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] \arrow[dd, "\pi"] & \\ \mathcal{Z}_1 \arrow[dr, "r_1"'] & & \mathcal{Z}_2 \arrow[dl, "r_2"] \\ & B & \end{tikzcd} \tag{4.4.3}$$

where p_1 and p_2, restricted to the preimage of a point $b \in B$, are the $\mathbb{P}^{r-1}$-bundle structures of the roof G/P, therefore they are $\mathbb{P}^{r-1}$-fibrations over $\mathcal{Z}_1$ and $\mathcal{Z}_2$.

For each homogeneous roof of the list (Kan18, Section 5.2.1), there exist homogeneous vector bundles E_1 and E_2 such that $\mathbb{P}(E_1) \simeq \mathbb{P}(E_2) \simeq G/P$. Hence, for $i = 1, 2$, they have the form:

$$E_i = G \times^{P_i} V_i \tag{4.4.4}$$

for a given representation space V_i. From the data of E_i we can define vector bundles on $\mathcal{Z}_i$ with the following construction:

$$\mathcal{E}_i := \mathcal{V} \times^G G \times^{P_i} V_i \tag{4.4.5}$$

Note that for every $b \in B$, we have $r_i^{-1}(b) \simeq G/P_i$ and $\mathcal{E}_i|_{r_i^{-1}(b)} \simeq E_i$. Since G/P is a roof, this implies that $(r_i^{-1}(b), \mathcal{E}_i|_{r_i^{-1}(b)})$ is a simple Mukai pair.

The line bundle $\mathcal{L}$ can be constructed in the following way: let us consider the line bundle L on G/P introduced in Proposition 4.1.4. Since L is homogeneous, there exists a one-dimensional representation space W such that $L = G \times^P W$. Let us fix:

$$\mathcal{L} := \mathcal{V} \times^G G \times^P W. \tag{4.4.6}$$

Such bundle restricts to L on every fiber of π, hence it restricts to $\mathcal{O}(1)$ on the fibers of both p_1 and p_2.

In order to prove the isomorphisms $\mathcal{Z} \simeq \mathbb{P}(\mathcal{E}_1) \simeq \mathbb{P}(\mathcal{E}_2)$ we proceed in the following way. Recall that, given a principal H-bundle $\mathcal{W}$ over a variety X there is the following exact functor (see (Nor82, Section 2.2), or the survey (Bal06, Page 8)):

$$H - \mathrm{Mod} \xrightarrow{\mathcal{W} \times^H (-)} \mathrm{Vect}(X) \tag{4.4.7}$$

which sends the H-module R to the vector bundle $\mathcal{W} \times^H R$ over X. Observe now that $\mathcal{V} \times^G G$ is a principal P-bundle over $\mathcal{Z}$, because $\mathcal{V} \longrightarrow \mathcal{V}/P$ is a principal P-bundle and $\mathcal{V}/P \simeq \mathcal{V} \times^G G/P \simeq \mathcal{Z}$ (see for example (Mit01, Proposition 3.5)). This allows to construct an exact functor:

$$P - \mathrm{Mod} \xrightarrow{\mathcal{V}\times^G G\times^P(-)} \mathrm{Vect}(\mathcal{Z}). \tag{4.4.8}$$

Let us now recall that there exists an equivalence of categories

$$\mathrm{Rep}(P) \xrightarrow{\quad F \quad} \mathrm{Vect}^P(G/P) \tag{4.4.9}$$

which sends the P-module H to the homogeneous vector bundle $G\times^P H$, hence F^{-1} is an exact functor. Summing all up we can construct an exact functor sending homogeneous vector bundles over G/P to vector bundles over $\mathcal{Z}$:

$$\begin{tikzcd}
\mathrm{Vect}^P(G/P) \arrow[rr, "\mathcal{V}\times^G G\times^P(-)\circ F^{-1}"] \arrow[dr, "F^{-1}"'] & & \mathrm{Vect}(\mathcal{Z}) \\
& P - \mathrm{Mod} \arrow[ur, "\mathcal{V}\times^G G\times^P(-)"'] &
\end{tikzcd} \tag{4.4.10}$$

By applying this functor to the following surjection (given by the relative Euler sequence of the projective bundle structure h_i)

$$h_i^* E_i \longrightarrow L \longrightarrow 0 \tag{4.4.11}$$

we get the following surjective map (compatibility with pullback follows from (Mit01, Proposition 3.6):

$$p_i^* \mathcal{E} \longrightarrow \mathcal{L} \longrightarrow 0 \tag{4.4.12}$$

which determines an isomorphism $\mathcal{Z} \longrightarrow \mathbb{P}(\mathcal{E})$ by (Har77, Ch. II.7, Proposition 7.12), and thus $p_{i*}\mathcal{L} \simeq \mathcal{E}$ by (Har77, Ch. II.7, Proposition

7.11).

Let us now prove that r_1 and r_2 are contractions of extremal rays. Observe that r_i is locally projective (hence proper) and every fiber is isomorphic to a Fano variety of Picard number one G/P_i, which means that $-K_{\mathcal{Z}_i}$ is r_i-ample because $-K_{\mathcal{Z}_i}|_{r_i^{-1}(b)}$ is ample for every $b \in B$ (Laz04a, Theorem 1.7.8). This proves that r_i is a Fano–Mori contraction (Occ99, Definition I.2.2). To prove that r_i is a contraction of extremal ray (or elementary Fano–Mori contraction) we just need to show that $\mathrm{Pic}(\mathcal{Z}_i)/\mathrm{Pic}(B) \simeq \mathbb{Z}$ (Occ99, Definition I.2.2). By (FI73, Proposition 2.3), since $\mathcal{Z}_i$ is a locally trivial G/P_i-fibration we have an exact sequence:

$$H^0(G/P_i, \mathcal{O}^*)/\mathbb{C}^* \longrightarrow \mathrm{Pic}(B) \longrightarrow \mathrm{Pic}(\mathcal{Z}_i) \longrightarrow \mathrm{Pic}(G/P_i) \longrightarrow 0 \tag{4.4.13}$$

and the first term vanishes because of the long cohomology sequence associated to the exponential sequence for G/P_i:

$$\begin{aligned} 0 \longrightarrow H^0(G/P_i, \mathbb{Z}) \longrightarrow H^0(G/P_i, \mathcal{O}) \longrightarrow \\ \longrightarrow H^0(G/P_i, \mathcal{O}^*) \longrightarrow H^1(G/P_i, \mathbb{Z}) \longrightarrow \cdots \end{aligned} \tag{4.4.14}$$

where we observe that $H^1(G/P_i, \mathbb{Z})$ vanishes since the integral cohomology of rational homogeneous varieties is generated by their Bruhat decompositon, and it is nonzero only in even degree. Our claim is proven once we recall that by Definition 4.1.2 one has $\mathrm{Pic}(G/P_i) \simeq \mathbb{Z}$.

□

Remark 4.4.4. Note that, for every $b \in B$, we have $r_i^{-1}(b) \simeq G/P_i$ and

$\mathcal{E}_i|_{r_i^{-1}(b)} \simeq E_i$. Since G/P is a roof, this implies that $(r_i^{-1}(b), \mathcal{E}_i|_{r_i^{-1}(b)})$ is a simple Mukai pair.

4.4.1 Calabi–Yau fibrations

One of our main interests is to investigate the zero loci of pairs of sections of $\mathcal{E}_1$ and $\mathcal{E}_2$ which are pushforwards of a section $\Sigma \in H^0(\mathcal{Z}, \mathcal{L})$, hence relativizing the setting of Definition 4.1.8. Let us make this clearer by the following lemma, the notation is established in Diagram 4.4.3.

Lemma 4.4.5. *Let $\mathcal{Z}$ be a homogeneous roof bundle of type $G/P \not\simeq \mathbb{P}^n \times \mathbb{P}^n$ over a smooth projective variety B and fix $h_i : G/P \simeq \mathbb{P}(E_i) \longrightarrow G/P_i$ for $i \in \{1; 2\}$. Suppose there exists a basepoint-free vector bundle $\mathcal{L}$ on $\mathcal{Z}$ such that for every $b \in B$ one has $\mathcal{L}|_{\pi^{-1}(b)} \simeq L$ and the restriction map $H^0(\mathcal{Z}, \mathcal{L}) \longrightarrow H^0(\pi^{-1}(b), L)$ is surjective. Given a general section $\Sigma \in H^0(\mathcal{Z}, \mathcal{L})$, let us call $X_i := Z(p_{i*}\Sigma)$. Then there exist fibrations:*

$$\begin{tikzcd} X_1 \arrow[dr, "f_1"'] & & X_2 \arrow[dl, "f_2"] \\ & B & \end{tikzcd} \tag{4.4.15}$$

such that for a general $b \in B$ the varieties $Y_1 := f_1^{-1}(b)$ and $Y_2 := f_2^{-1}(b)$ are a Calabi–Yau pair associated to the roof G/P in the sense of Definition 4.1.8.

Proof. Since $p_{i*}\mathcal{L} = \mathcal{E}_i$, $X_i \subset \mathcal{Z}_i$ is the zero locus of a section $p_{i*}\Sigma$ of $\mathcal{E}_i$. Let us call $f_i := r_i|_{X_i}$. By the condition $\mathcal{E}_i|_{r_i^{-1}(b)} \simeq E_i$ and

$r_i^{-1}(b) \simeq G/P_i$ it follows that $(r_i^{-1}(b), \mathcal{E}_i|_{r_i^{-1}(b)})$ is a Mukai pair. If b and Σ are general the varieties $Y_i = Z(p_{i*}\Sigma|_{r_i^{-1}(b)}) \subset r_i^{-1}(b)$ are Calabi–Yau by Lemma 2.5.3 and the fact that the general Σ has smooth zero locus. Moreover, $E_i \simeq h_{i*}L$ and the varieties Y_1 and Y_2 are the zero loci of the pushforwards of the same section $\Sigma_{\pi^{-1}(b)}$, therefore they are a Calabi–Yau pair associated to the roof of type G/P as in Definition 4.1.8.

□

4.5 Example: a pair of Calabi–Yau eightfolds

4.5.1 Roof of type A_4^G

We briefly recall a description of the roof of type A_4^G and its related dual Calabi–Yau threefolds. Let V_5 be a vector space of dimension five. We call $G(2, V_5)$ and $G(3, V_5)$ the Grassmannians of respectively linear 2-spaces and linear 3-spaces in V_5. On such Grassmannians, there are the following universal (tautological) short exact sequences:

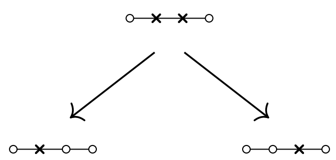
$$0 \longrightarrow \mathcal{U}_{G(2,V_5)} \longrightarrow V_5 \otimes \mathcal{O} \longrightarrow \mathcal{Q}_{G(2,V_5)} \longrightarrow 0 \tag{4.5.1}$$

$$0 \longrightarrow \mathcal{U}_{G(3,V_5)} \longrightarrow V_5 \otimes \mathcal{O} \longrightarrow \mathcal{Q}_{G(3,V_5)} \longrightarrow 0 \tag{4.5.2}$$

where $\det \mathcal{U}^\vee_{G(2,V_5)} \simeq \det \mathcal{Q}_{G(2,V_5)} \simeq \mathcal{O}_{G(2,V_5)}(1)$ and $\det \mathcal{U}^\vee_{G(3,V_5)} \simeq \det \mathcal{Q}_{G(3,V_5)} \simeq \mathcal{O}_{G(3,V_5)}(1)$. The flag variety $F(2, 3, V_5)$ admits two projective bundle structures, which define projections to the Grassmannians. These data define the roof of type A_4^G, illustrated by the following diagram:

$$
\begin{tikzcd}
\mathbb{P}\mathcal{Q}^\vee_{G(2,V_5)}(2) \arrow[d, "h_1"] \arrow[r, equal] & F(2,3,V_5) \arrow[r, equal] & \mathbb{P}\mathcal{U}_{G(3,V_5)}(2) \arrow[d, "h_2"'] \\
G(2,V_5) & & G(3,V_5)
\end{tikzcd}
\tag{4.5.3}
$$

There exists a line bundle L on $F(2,3,V_5)$ such that $h_{1*}L \simeq \mathcal{Q}^\vee_{G(2,V_5)}(2)$ and $h_{2*}L \simeq \mathcal{U}_{G(3,V_5)}(2)$. Zero loci of sections of such pushforwards are Calabi–Yau threefolds. Moreover, for a general $S \in H^0(F(2,3,V_5),L)$, the pushforwards $h_{1*}S$ and $h_{2*}S$ are a pair of non isomorphic (and hence non birational) derived equivalent Calabi–Yau threefolds (see Theorem 7.2.6 for non birationality, and Section 10.5 for derived equivalence). The roof of type A_4^G can be described by the following Dynkin diagrams:



(4.5.4)

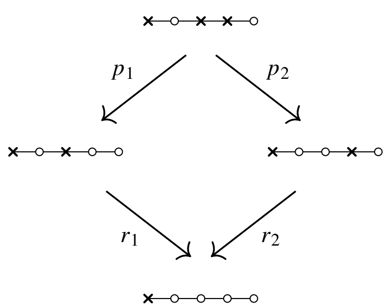
where we described a quotient G/P by the crossed Dynkin diagram corresponding to P (see Chapter 2). Observe that, in the basis of fundamental weights $\{\omega_1,\dots,\omega_4\}$, L is the homogeneous line bundle whose weight is $\omega_2+\omega_3$, we write $L = \mathcal{E}_{\omega_2+\omega_3}$. Given a dominant weight ω, we denote V_ω the associated representation space. By the Borel–Weil–Bott theorem we have $H^0(F(2,3,V_5),L) \simeq H^0(G(2,V_5),\mathcal{Q}^\vee_{G(2,V_5)}(2)) \simeq H^0(G(3,V_5),\mathcal{U}_{G(3,V_5)}(2)) \simeq V_{\omega_2+\omega_3}$ which is a 75-dimensional vector space.

4.5.2 The homogeneous roof bundle of type $SL(5)/P^{2,3}$ over $\mathbb{P}^5$

Let us fix a vector space $V_6 \simeq \mathbb{C}^6$ and the (twisted) tangent bundle $T(-1)$ of $\mathbb{P}(V_6) \simeq \mathbb{P}^5$. As in Remark 4.4.2, we can define a roof bundle of type $SL(5)/P^{2,3}$ over $\mathbb{P}^5$ by considering the $SL(5)$-bundle $\mathcal{V} = \mathrm{Iso}(\mathbb{C}^5, T(-1))$ and taking $\mathcal{V} \times^{SL(5)} (SL(5)/P^{2,3}) \simeq \mathcal{F}l(2,3,T(-1))$. In this setting, Diagram 4.4.3 becomes:

$$\begin{tikzcd}
 & \mathcal{F}l(2,3,T(-1)) \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
\mathcal{G}r(2,T(-1)) \arrow[dr, "r_1"'] & & \mathcal{G}r(3,T(-1)) \arrow[dl, "r_2"] \\
 & \mathbb{P}^5 &
\end{tikzcd} \tag{4.5.5}$$

Note that $G(1,V_6) \simeq \mathbb{P}^5$ is a homogeneous variety and the whole construction can be sketched in terms of crossed Dynkin diagrams:



(4.5.6)

This picture is obtained extending the Dynkin diagrams of Diagram 4.5.4 with a new crossed root from the left, the notation is the same of

Diagram 4.5.4.

The associated varieties are respectively $F(1,3,4,V_6)$, $F(1,3,V_6)$, $F(1,4,V_6)$ and $G(1,V_6)$, hence Diagram 4.5.5 can be rewritten as:

$$\begin{tikzcd}
 & & F(1,3,4,V_6) \arrow[dl, "p_1"'] \arrow[dd, "\pi"] \arrow[dr, "p_2"] & & \\
 & F(1,3,V_6) \arrow[dl, "\rho"'] \arrow[dr, "r_1"] & & F(1,4,V_6) \arrow[dl, "r_2"'] \arrow[dr, "\tau"] & \\
G(3,V_6) & & G(1,V_6) & & G(4,V_6)
\end{tikzcd} \tag{4.5.7}$$

Here r_1 and r_2 are Grassmannian bundles, where the fibers are identified respectively with $G(2,V_5)$ and $G(3,V_5)$. Moreover, one has the $\mathbb{P}^2$-bundle $\rho : F(1,3,V_6) \longrightarrow G(3,V_6)$ and the $\mathbb{P}^3$-bundle $\tau : F(1,4,V_6) \longrightarrow G(4,V_6)$.

In the following, given a weight ω, we will call $\mathcal{E}_\omega$ the associated vector bundle. Given a dominant weight ω, we will call V_ω the associated representation space. On $F(1,3,4,V_6)$ we use the following notations for line bundles: $\mathcal{O}(a,b,c) := \pi^*\mathcal{O}(a) \otimes p_1^*\rho^*\mathcal{O}(b) \otimes p_2^*\tau^*\mathcal{O}(c)$. Fix the standard basis $\{\omega_1, \dots, \omega_5\}$ of fundamental weights for A_5.

Observe that $\mathcal{O}(1,1,1) = \mathcal{E}_{\omega_1+\omega_3+\omega_4}$ on $F(1,3,4,V_6)$ has pushforwards to the Picard rank 2 flag varieties given by $p_{1*}\mathcal{O}(1,1,1) = \rho^*\mathcal{Q}^\vee_{G(3,V_6)}(1,2)$ and $p_{2*}\mathcal{O}(1,1,1) = \mathcal{P}(1,2)$ where $\mathcal{P}$ is the rank

3 vector bundle defined by the following short exact sequence on $F(1,4,V_6)$:

$$0 \longrightarrow \mathcal{O}(-1,0) \longrightarrow \tau^*\mathcal{U}_{G(4,V_6)} \longrightarrow \mathcal{P} \longrightarrow 0. \tag{4.5.8}$$

4.5.3 A pair of Calabi–Yau eightfolds

Lemma 4.5.1. *Let $S \in H = H^0(F(1,3,4,V_6),\mathcal{O}(1,1,1))$ be a general section. Then $X_1 = Z(p_{1*}S)$ and $X_2 = Z(p_{2*}S)$ are Calabi–Yau eightfolds of Picard number 2, and $H^1(X_i,T_{X_i}) \simeq H/(\mathbb{C}\oplus V_{\omega_1+\omega_5}) \simeq \mathbb{C}^{1014}$. Moreover, there exist fibrations $f_i : X_i \longrightarrow \mathbb{P}(V_6)$ such that for the general $b \in B$ the pair $(f_1^{-1}(b), f_2^{-1}(b))$ is a Calabi–Yau pair associated to the roof of type A_4^G.*

Proof. As above, fix the shorthand notation $\mathcal{E}_i := p_{i*}\mathcal{O}(1,1,1)$. One has $\det(\mathcal{Q}^\vee_{G(3,V_6)}(1,2)) = \mathcal{O}(3,5)$ on $F(1,3,V_6)$ and $\det(\mathcal{P}(1,2)) \simeq \mathcal{O}(4,5)$ on $F(1,4,V_6)$, while $\omega_{F(1,3,V_6)} \simeq \mathcal{O}(-3,-5)$ and $\omega_{F(1,4,V_6)} \simeq \mathcal{O}(-4,-5)$. Hence, sections of $\mathcal{E}_i$ define eight dimensional varieties with vanishing first Chern class for $i \in \{1;2\}$. Since the Grothendieck line bundle of $\mathbb{P}(\mathcal{E}_i)$ is an ample line bundle, $\mathcal{E}_i$ is an ample vector bundle and we can use again (Laz04b, Example 7.1.5): the restriction maps

$$\begin{aligned} H^q(F(1,3,V_6),\Omega^p_{F(1,3,V_6)}) &\longrightarrow H^q(X_1,\Omega^p_{X_1}) \\ H^q(F(1,4,V_6),\Omega^p_{F(1,4,V_6)}) &\longrightarrow H^q(X_2,\Omega^p_{X_2}) \end{aligned} \tag{4.5.9}$$

are isomorphisms for $p+q < \dim(X_1)$, and since $F(1,3,V_6)$ and $F(1,4,V_6)$ are homogeneous varieties, their structure sheaves have cohomology of dimension one concentrated in degree zero. The Calabi–

Yau condition follows from setting $p = 0$ in the isomorphism of Equation 4.5.9.

In order to compute cohomology for the tangent bundle, let us first focus on X_1. We consider the following two projections:

$$\begin{tikzcd} & F(1,3,V_6) \arrow[dl, "r_1"'] \arrow[dr, "\rho"] & \\ G(1,V_6) & & G(3,V_6) \end{tikzcd} \tag{4.5.10}$$

and the following exact sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow \rho^*\mathcal{U}(1,-1) \longrightarrow T_{F(1,3,V_6)} \longrightarrow \rho^* T_{G(3,V_6)} \longrightarrow 0 \tag{4.5.11}$$

which follows by the relative tangent bundle sequence of $F(1,3,V_6) \longrightarrow G(3,V_6)$ and the relative Euler sequence of the projective bundle structure $F(1,3,V_6) \simeq \mathbb{P}(r^*\mathcal{U}(1,-1))$.

By the Borel–Weil–Bott theorem we get

$$H^m(X, T_X) \simeq \begin{cases} V_{\omega_1+\omega_3+\omega_4}/(\mathbb{C} \oplus V_{\omega_1+\omega_5}) & m = 1 \\ \\ \mathbb{C}^2 & m = 7 \end{cases} \tag{4.5.12}$$

and this proves our claim. In fact, since Y is Calabi–Yau, by Serre duality we have:

$$H^7(Y, T_Y) \simeq H^1(Y, \Omega^1_Y) = H^{(1,1)}(Y) \tag{4.5.13}$$

and we conclude that the Picard number of Y is two by the long exact sequence of cohomology of the exponential sequence.

The case of X_2 is identical: in fact, the sequence of Equation 4.5.11 involves only bundles on $F(1,3,V_6)$, and the weights of the bundles involved in the corresponding sequence on $F(1,4,V_6)$ are obtained by reversing the order of the fundamental weights on the crossed Dynkin diagram of the flag variety. Therefore, the result is the same by the symmetry of the Dynkin diagram of type A_5.

The proof is concluded by observing that since $\mathcal{O}(1,1,1)$ is basepoint-free and the restriction

$$H^0(F(1,3,4,6),\mathcal{O}(1,1,1)) \longrightarrow H^0(F(2,3,5),\mathcal{O}(1,1))$$

is surjective, we can apply Lemma 4.4.5. $\square$

5 $\mathbb{L}$-equivalence

5.1 The Grothendieck ring of varieties

Definition 5.1.1. *Let S be the additive group of formal linear combinations of complex algebraic varieties $\sum a_i[Y_i]$ with integral coefficient. The Grothendieck ring of complex varieties $K_0(Var/\mathbb{C})$ is the quotient of S by respect to the following equivalence relation:*

$$[Y_1] - [Y_2] - [Y_1 \setminus Y_2] \tag{5.1.1}$$

and a ring structure defined by the Cartesian product:

$$[Y_1] \cdot [Y_2] = [Y_1 \times Y_2] \tag{5.1.2}$$

In the remainder of this work, since we are only interested in complex varieties, we will refer to such object as "the Grothendieck ring" without ambiguity.

One has the following standard result (see for example (CLNS18, Proposition 2.3.3)):

Lemma 5.1.2. *Let $f : X \longrightarrow Y$ be a piecewise trivial fibration with fiber F. Then one has $[X] = [Y] \cdot [F]$.*

By Equation 5.1.2 we see that the class of a point is $[pt] = 1$. If we call $\mathbb{L}$ the class of the affine line, the following is a standard result (e.g. (CLNS18, Example 2.4.1)):

$$[\mathbb{P}^n] = 1 + \mathbb{L} + \cdots + \mathbb{L}^n. \tag{5.1.3}$$

Lemma 5.1.3. *(GS14, Lemma 2.1) Let Y_1 and Y_2 be smooth birationally equivalent varieties of dimension n. Then there exists the following equality in the Grothendieck ring of varieties:*

$$[Y_1] - [Y_2] = \mathbb{L} \cdot \mathcal{M} \tag{5.1.4}$$

where $\mathcal{M}$ is a linear combination of classes of smooth projective varieties of dimension $n - 2$.

Corollary 5.1.4. *(GS14, Corollary 2.2) If Y is a rational smooth d-dimensional variety, then:*

$$[Y] = [\mathbb{P}^n] + \mathbb{L} \cdot \mathcal{M}_Y \tag{5.1.5}$$

where $\mathcal{M}_Y$ is a linear combination of classes of smooth projective varieties of dimension $n - 2$.

The class in the Grothendieck ring is a powerful birational invariant, appearing in many useful applications. Let us remind the notion of *stably birational equivalence* and its relation with the more common concept of birational equivalence:

Definition 5.1.5. *Two varieties X and Y are* stably birational *if there exist integers $m, n \geq 0$ such that $X \times \mathbb{P}^m$ and $Y \times \mathbb{P}^n$ are biratinally equivalent.*

This condition is weaker than the usual notion of birationally equivalent. For instance, while $X = \mathbb{P}^1$ and $Y = \{pt\}$ are of course not birational, they are stably birational. However, one has the following fact:

Lemma 5.1.6. *(GS14, Lemma 2.6) Let X and Y are stably birational varieties of the same dimension. If X is not uniruled, then X and Y are birational.*

Theorem 5.1.7. *(LL03) Let $\mathcal{I} \subset K_0(Var/\mathbb{C})$ be the ideal generated by the class of the affine line. One has the following ring isomorphism:*

$$K_0(Var/\mathbb{C})/\mathcal{I} \simeq \mathbb{Z}[\mathcal{SB}] \tag{5.1.6}$$

where $\mathcal{SB}$ is the multiplicative monoid of classes of stable birational equivalence of varieties.

Corollary 5.1.8. *(LL03, Corollary 2.6) Let $X_1, \ldots, X_m, Y_1, \ldots, Y_n$ be smooth complete varieties such that they satisfy the following relation in the Grothendieck ring:*

$$\sum_{i=1}^{m} a_i[X_i] = \sum_{j=1}^{n} b_j[Y_j] \tag{5.1.7}$$

for some coefficients $a_1, \ldots, a_m, b_1, \ldots, b_n$. Then the following holds:

- $m = n$
- *Up to reordering, X_i and Y_i are stably birational for every $i \leq n$*
- $a_i = b_i$ *for every* $i \leq n$

A consequence of this fact is that the class of any variety, up to stable birationality, is a unique linear combination of classes of smooth complete varieties. Heuristically, this provides a sort of "basis" of the Grothendieck ring in terms of classes of smooth complete varieties.

The notion of $\mathbb{L}$-equivalence became popular in the context of the problem of rationality of cubic hypersurface. In fact, Galkin and Shinder proved the following result:

Theorem 5.1.9. *(GS14, Theorem 7.4) Let k be a field such that $\mathbb{L} \in K_0(Var/k)$ is not a zero divisor. Then, any smooth cubic threefold Y/k is irrational.*

This result motivated the search for non birationally equivalent pairs of complex varieties such that the (nonzero) difference of their classes in the Grothendieck ring annihilates the class of the affine line, or a higher power $\mathbb{L}^m$. This motivates the following definition:

Definition 5.1.10. *((KS18, Section 1.2)) Let X, Y be complex varieties. We say that X and Y are $\mathbb{L}$-equivalent if the following equation holds for some integer $m \geq 0$:*

$$([X] - [Y])\mathbb{L}^m = 0 \tag{5.1.8}$$

Examples of $\mathbb{L}$-equivalent pairs which are not birationally equivalent have been found among Calabi–Yau varieties: the first is given by the Pfaffian–Grassmannian pair (BC08), already studied in a physical context by (Rød98). All pairs of non birational and $\mathbb{L}$ equivalent Calabi–Yau varieties have the property of being derived equivalent. Understanding the link between derived equivalence and $\mathbb{L}$-equivalence and giving an interpretation to the minimal exponent m associated to an $\mathbb{L}$-equivalence, are still open problems.

Conjectures have been formulated in the past decade: for instance, Kuznetsov and Shinder formulated a conjecture such that for smooth projective simply connected varieties, derived equivalence implies $\mathbb{L}$-equivalence (KS18, Conjecture 1.6).

Remark 5.1.11. The assumption of simply connectedness of (KS18, Conjecture 1.6) is needed to rule out the case of abelian varieties, for which an example of a derived equivalent pair which is not $\mathbb{L}$-equivalent has been given by Efimov (Efi18).

Examples of pairs of derived equivalent and (nontrivially) $\mathbb{L}$-equivalent Calabi–Yau varieties and $K3$ surfaces have been object of a recent series of papers (IMOU19; Kuz18; KR17; KR20; KKM20; HL18; OR17; Man17; KS18).

5.2 $\mathbb{L}$-equivalence and Calabi–Yau pairs associated to a roof

Let us consider a roof $X \simeq \mathbb{P}(\mathcal{E}_1) \simeq \mathbb{P}(\mathcal{E}_2)$, where for $i \in \{1; 2\}$ one has the structure morphisms $h_i : \mathbb{P}(\mathcal{E}_i) \longrightarrow B_i$. Consider a section σ of L with smooth zero locus, where L is the line bundle which restricts to $\mathcal{O}(1)$ on every fiber of h_1 and h_2, see Proposition 4.1.4 for details. Call (Y_1, Y_2) the associated Calabi–Yau pair in the sense of Definition 4.1.8. We get the following diagram:

$$
\begin{tikzcd}
& T_1 \arrow[r, hook, "k_1"] \arrow[ddl, "\nu_1"'] & M \arrow[d, hook, "l"] \arrow[ddl, "\bar{h}_1"'] \arrow[ddr, "\bar{h}_2"] & T_2 \arrow[l, hook', "k_2"'] \arrow[ddr, "\nu_2"] & \\
& & X \arrow[dl, "h_1"] \arrow[dr, "h_2"'] & & \\
Y_1 \arrow[r, hook, "t_1"] & B_1 & & B_2 & Y_2 \arrow[l, hook', "t_2"']
\end{tikzcd}
\tag{5.2.1}
$$

where M is the zero locus of σ, $\bar{h}_i$ the restriction of h_i to M, T_i is the preimage of Y_i under $\bar{h}_i$ and ν_i is the restriction of $\bar{h}_i$ to T_i.

We observe that the fibers of the surjections $\bar{h}_i$ have the following description:

$$\bar{h}_i^{-1}(y) \simeq \begin{cases} \mathbb{P}^{r-1} & \text{if } y \in Y_i \\ \mathbb{P}^{r-2} & \text{if } y \in B_i \setminus Y_i \end{cases} \tag{5.2.2}$$

due to the fact that Y_i is the zero locus of the pushforward of S to B_i, and T_i is isomporphic to the projectivization of the normal bundle $\mathcal{E}_i|_{Y_i}$ of Y_i in B_i.

Let us consider the roof from the point of view of the Grothendieck ring of varieties. First of all observe that the bases B_1 and B_2 are stably birational, therefore by results of (LL03) have equal class in the quotient of the Grothendieck ring by the ideal generated by the $\mathbb{L}$.

However, if we assume the stronger condition that B_1 and B_2 have equal class in the Grothendieck ring we get an interesting consequence, namely the difference of the classes of a pair of Calabi–Yau varieties associated to the roof annihilates the $r-1$-th power of the class of the affine line.

In light of Lemma 5.1.2, we begin by showing that the maps $\bar{h}_i$ are piecewise trivial fibrations in the sense of (CLNS18, Definition 2.3.1). This is a consequence of the fact that $\bar{h}_i^{-1}(Y_i) = \mathbb{P}(\mathcal{E}|_{Y_i})$, and that $\bar{h}_i^{-1}(B \setminus Y_i) = \mathbb{P}((\mathcal{E}|_{B\setminus Y_i}/(h_{i*}S|_{B\setminus Y_i} \otimes \mathcal{O}_B|_{B\setminus Y_i})))$ for $i \in \{1;2\}$.
Now, let M be the hyperplane section of X defining the pair. Then we

have the following two descriptions of the class of M:

$$[M] = [\mathbb{P}^{r-1}] \cdot [Y_1] + [\mathbb{P}^{r-2}] \cdot [B_1 \setminus Y_1]$$

$$[M] = [\mathbb{P}^{r-1}] \cdot [Y_2] + [\mathbb{P}^{r-2}] \cdot [B_2 \setminus Y_2]$$

By the relations defining the Grothendieck ring of varieties we have $[B_1 \setminus Y_1] = [B_1] - [Y_1]$, and the same holds for the second equation. Then, subtracting the two equations above, we get

$$\begin{aligned} 0 &= [\mathbb{P}^{r-1}] \cdot [Y_1] + [\mathbb{P}^{r-2}] \cdot ([B_1] - [Y_1]) + \\ &\quad - [\mathbb{P}^{r-1}] \cdot [Y_2] + [\mathbb{P}^{r-2}] \cdot ([B_2] - [Y_2]) \\ &= ([\mathbb{P}^{r-1}] - [\mathbb{P}^{r-2}])([Y_1] - [Y_2]) + [\mathbb{P}^{r-2}] \cdot ([B_1] - [B_2]). \end{aligned}$$

Given the identity

$$[\mathbb{P}^k] = 1 + \mathbb{L} + \mathbb{L}^2 + \cdots \mathbb{L}^k.$$

we have the following formula, developed in (KR20):

$$([Y_1] - [Y_2])\mathbb{L}^{r-1} + [\mathbb{P}^{r-2}] \cdot ([B_1] - [B_2]) = 0 \tag{5.2.3}$$

Remark 5.2.1. In most of the known examples of roofs, Equation 5.2.3 provides $\mathbb{L}$-equivalence for the associated Calabi–Yau pairs because the bases B_1 and B_2 of the roof are isomorphic. This is the case of roofs of type A_{2k}, D_k and $G_2^\dagger$. For the roof of type G_2, it has been proved that $[B_i] = [\mathbb{P}^5]$ in (IMOU19), and $\mathbb{L}$-equivalence follows.

6 Hodge structures

In this chapter we review a generalization of the blowup formula for cohomology in order to compare Hodge structures in the algebraic middle cohomologies of Calabi-Yau pairs associated to a roof. Let us begin by stating the classical result for blowups.

Let $\widetilde{X}$ be the blowup of a smooth projective variety X in a smooth subvariety Y, and let E be the exceptional divisor. We get the following diagram:

$$\begin{tikzcd} E \arrow[r, hook, "j"] \arrow[d, "p"] & \widetilde{X} \arrow[d, "\tau"] \\ Y \arrow[r, hook, "i"] & X \end{tikzcd} \tag{6.0.1}$$

The Hodge structure of $\widetilde{X}$ in terms of the ones of Y and X can be computed by means of the following standard result:

Theorem 6.0.1. *(Voi10, Theorem 7.31) Let X be a Kähler manifold, and let $Y \subset X$ be a submanifold of codimension r. Call $\tau : \widetilde{X} \longrightarrow X$ the blowup of X in Y and E the exceptional divisor. Then, one has the following isomorphism of Hodge structures:*

$$\bigoplus_{i=0}^{r-2} H^{k-2i-2}(Y,\mathbb{Z})(i+1,i+1) \oplus H^k(X,\mathbb{Z}) \xrightarrow{\sim} H^k(\widetilde{X},\mathbb{Z}) \tag{6.0.2}$$

which acts on classes in the following way:

$$x_0, \dots x_{r-2}, z \longrightarrow j_* p^* x_0 + j_*((p^* x_1 \cup \xi) + \cdots + j_*(p^* x_{r-2} \cup \xi^{r-2}) + \tau^* z) \tag{6.0.3}$$

6.1 Cohomological Cayley trick

Let X be a roof with projective bundle structures $h_i : X \longrightarrow Z_i$ for $i \in \{1; 2\}$, and let (Y_1, Y_2) be a Calabi–Yau pair associated to a section of L with zero locus $M \subset X$, in the sense of Definition 4.1.8. The scope of this section is to give a dual description of the cohomology of M in terms, respectively, of the cohomologies of Y and Z_1 and the cohomologies of Y_2 and Z_2.

More generally, given a Mukai pair $(\mathcal{E}, Z)$ and a hyperplane section $M \subset X \simeq \mathbb{P}\mathcal{E}$, we establish the following diagram, which will be the setting of Theorem 6.1.1:

$$\begin{tikzcd} p^{-1}(Y) \arrow[r, hook, "j"] \arrow[d, "q"] & M \arrow[r, hook] \arrow[d, "p"] & X \arrow[dl, "\pi"] \\ Y \arrow[r, hook] & Z & \end{tikzcd} \tag{6.1.1}$$

where p is a fibration such that:

$$p^{-1}(x) \simeq \begin{cases} \mathbb{P}^{r-1} & x \in Y \\ \mathbb{P}^{r-2} & x \in Z \setminus Y \end{cases} \tag{6.1.2}$$

and q is the restriction of p to the preimage $p^{-1}(Y)$. Note that this is the same situation occurring for the maps $\bar{h}_1$ and $\bar{h}_2$ in Diagram 5.2.1.

Theorem 6.1.1 (Cohomological Cayley trick)**.** *Let Z be a Kähler manifold of dimension n, let $\mathcal{E}$ be a vector bundle of rank r over Z and $X = \mathbb{P}(\mathcal{E}^\vee)$. Then, given a smooth hyperplane section $M = Z(S) \subset X$*

and a section $Y = Z(\pi_* S) \subset Z$ smooth of codimension r, there exists an isomorphism of Hodge structures

$$\Xi : \bigoplus_{i=0}^{r-2} H^{k-2i}(Z,\mathbb{Z}) \oplus H^{k-2r+2}(Y,\mathbb{Z}) \xrightarrow{\sim} H^k(M,\mathbb{Z}) \tag{6.1.3}$$

which acts on classes in the following way:

$$x_0, \dots x_{r-2}, z \longrightarrow p^* x_0 + p^* x_1 \cup \xi + \cdots + p^* x_{r-2} \cup \xi^{r-2} + j_* q^* z \tag{6.1.4}$$

where $\xi \in H^2(M,\mathbb{Z})$ is the restriction to M of the hyperplane class of X related to the Grothendieck line bundle $\mathcal{O}_X(1)$.

Proof. The theorem is part of mathematical folklore. Its proof is analogous to the proof of the blowup formula (Voi10, Theorem 7.31) and now contained in the recent preprint (BFM19, Proposition 46), and a different proof is given in (DK20). □

6.2 The middle cohomology

Let us specialize to $k = n + r - 2$. Then Theorem 6.1.1 gives the following morphism of middle cohomologies:

$$H^{n-r}(Y,\mathbb{Z}) \xhookrightarrow{\Xi|_{H^{n-r}(Y,\mathbb{Z})} = j_* \circ q^*} H^{n+r-2}(M,\mathbb{Z}) \tag{6.2.1}$$

Note also that

$$\Xi\left(\bigoplus_{i=0}^{r-2} H^{n+r-2-2i}(Z,\mathbb{Z})\right) \perp j_* \circ q^*(H^{n-r}(Y,\mathbb{Z})) \tag{6.2.2}$$

where $\perp$ is taken with respect to the cup product in $H^{n+r-2}(M,\mathbb{Z})$. Indeed, this follows from dimensional reasons since $x_i|_Y \cup H^{n-r}(Y,\mathbb{Z}) = 0$ for $x_i \in H^{n+r-2-2i}(Z,\mathbb{Z})$ where $i \le r-2$.

Furthermore, we claim that $j_* \circ q^*$ preserves the cup product pairing up to a sign determined by the rank of $\mathcal{E}$.

Lemma 6.2.1. *For every $D_1, D_2 \in H^{n-r}(Y,\mathbb{Z})$, the map $j_* \circ q^*$ of Equation* 6.2.1 *satisfies the following identity, where r is the rank of $\mathcal{E}$:*

$$(D_1 \cdot D_2)_Y = (-1)^{r-1}(j_*q^*D_1 \cdot j_*q^*D_2)_M. \tag{6.2.3}$$

Proof. Let us work on the right-hand side of Equation 6.2.3: by an application of the projection formula we have

$$(j_*q^*D_1 \cdot j_*q^*D_2)_M = j_*(j^*j_*q^*D_1 \cdot q^*D_2)_{p^{-1}(Y)}.$$

Let us focus on the term $j^*j_*q^*D_1$. By the self-intersection formula ((LMS75, Theorem 1)) we have:

$$j^*j_*q^*D_1 = q^*D_1 \cdot c_{r-1}(\mathcal{N}_{p^{-1}(Y)|M}).$$

Substituting this in the main equation we get

$$(j_*q^*D_1 \cdot j_*q^*D_2)_M = j_*(q^*D_1 \cdot c_{r-1}(\mathcal{N}_{p^{-1}(Y)|M}) \cdot q^*D_2)_{p^{-1}(Y)}. \tag{6.2.4}$$

The class $c_{r-1}(\mathcal{N}_{p^{-1}(Y)|M}) \in H^{2r-2}(M;\mathbb{Z})$ can be described in terms of the class ξ of the Grothendieck line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E}^\vee)}(1)$ and the generators C_i of the cohomology ring of Z as

$$[p^{-1}(Y)]_M = a\xi^{r-1} + \sum b_i \xi^i C_i$$

with $a, b_i \in \mathbb{Z}$. Since the only contributing term of $c_{r-1}(\mathcal{N}_{p^{-1}(Y)|M})$ in Equation 6.2.4 is $a\xi^{r-1}$, the proof reduces to showing that $a = (-1)^{r-1}$. This can be done observing that

$$a = \deg c_{r-1}(\mathcal{N}_{p^{-1}(Y)|M}|_F)$$

where F is the fiber of M over a point in Y, and it is isomorphic to $\mathbb{P}^{r-1}$. By the following sequence of normal bundles

$$0 \longrightarrow \mathcal{N}_{p^{-1}(Y)|M} \longrightarrow \mathcal{N}_{p^{-1}(Y)|X} \longrightarrow \mathcal{O}_X(1)|_{p^{-1}(Y)} \longrightarrow 0$$

and by the fact that the restriction of $\mathcal{N}_{p^{-1}(Y)|X}$ to F is trivial, we get $a = 1$ if r is odd, and $a = -1$ otherwise. □

Suppose now that $H^*(Z, \mathbb{Z})$ is algebraic (which holds for example for rational homogeneous varieties), then the only non-algebraic part of the middle cohomology of M comes from $H^{n-r}(Y, \mathbb{Z})$. More precisely, since $j_* \circ q^*$ and p^* map algebraic classes to algebraic classes, we have

$$H^{n+r-2}_{alg}(M, \mathbb{Z}) = \bigoplus_{i=0}^{r-2} H^{n+r-2-2i}(Z, \mathbb{Z}) \oplus H^{n-r}_{alg}(Y, \mathbb{Z}).$$

Indeed, the sum of algebraic classes in the right hand side is algebraic, but also whenever we take an algebraic class in M and decompose it by means of (6.1.3), since all its components from $\bigoplus_{i=0}^{r-2} H^{n+r-2-2i}(Z, \mathbb{Z})$ are algebraic by assumption then also the component in $H^{n-r}(Y, \mathbb{Z})$ must be algebraic.

Moreover, if we define $T_Y := H^{n-r}_{alg}(Y, \mathbb{Z})^{\perp} \subset H^{n-r}(Y, \mathbb{Z})$ and $T_M := H^{n+r-2}_{alg}(M, \mathbb{Z})^{\perp} \subset H^{n+r-2}(M, \mathbb{Z})$, by Equation 6.2.2 we also have $j_* \circ q^*(T_Y) = T_M$ and by Lemma 6.2.1 $j_* \circ q^*$ defines an isometry between these lattices up to a sign depending on the rank of $\mathcal{E}$. The results of Theorem 6.1.1 and Lemma 6.2.1 apply for each side of a roof diagram like Diagram 5.2.1. Hence, we have two maps Ξ_1 and Ξ_2. Now, provided that both Z_1 and Z_2 have algebraic cohomology, we have the following isomorphism of integral Hodge structures:

$$\begin{array}{c} \bigoplus_{i=0}^{r-2} H^{k-2i}(Z_1,\mathbb{Z}) \oplus H^{k-2r+2}(Y_1,\mathbb{Z}) \\ \Big\downarrow{\scriptstyle \Xi_2^{-1}\circ\Xi_1} \\ \bigoplus_{i=0}^{r-2} H^{k-2i}(Z_2,\mathbb{Z}) \oplus H^{k-2r+2}(Y_2,\mathbb{Z}) \end{array} \tag{6.2.5}$$

which by Theorem 6.1.1, Lemma 6.2.1, Equation 6.2.2 and (Huy16, Lemma 3.1), after restriction to T_Y determines a Hodge isometry $T_{Y_1} \simeq T_{Y_2}$ defined by $(\widetilde{j}_* \circ \widetilde{q}^*|_{T_{Y_2}})^{-1} \circ j_* \circ q^*|_{T_{Y_1}}$. In light of the derived global Torelli theorem for $K3$ surfaces (Orl97, Theorem 3.3), this Hodge isometry gives us information about the derived categories of Calabi–Yau pairs associated to a roof with $n - r = 2$. This will described more precisely in Chapter 8.

Another application of Lemma 6.2.1 and Theorem 6.1.1 is following proposition:

Proposition 6.2.2. *Let $X \simeq \mathbb{P}(\mathcal{E}_1^\vee) \simeq \mathbb{P}(\mathcal{E}_2^\vee)$ be a roof of dimension $2r + 2k$, where $\mathcal{E}_1$ and $\mathcal{E}_2$ are vector bundles of rank r such that their bases Z_1 and Z_2 have no odd-degree integral cohomology. Let $(Y, \widetilde{Y})$ be the Calabi–Yau pair associated to X defined by a section of $\mathcal{L}$ with smooth* *zero locus $M \subset X$. Then there exists a Hodge isometry $H^{2k+1}(Y,\mathbb{Z}) \simeq H^{2k+1}(\widetilde{Y},\mathbb{Z})$.*

Proof. In this setting, Theorem 6.1.1 defines the following isomorphism:

$$\bigoplus_{i=0}^{r-2} H^{2r+2k-1-2i}(Z,\mathbb{Z}) \oplus H^{2k+1}(Y,\mathbb{Z}) \xrightarrow{\sim} H^{2r+2k-1}(M,\mathbb{Z}) \tag{6.2.6}$$

where all the summands $H^{2r+2k+1-2i}(Z,\mathbb{Z})$ are trivial. Then the proof follows from Lemma 6.2.1. □

Remark 6.2.3. Proposition 6.2.2 applies to all known examples of roofs where $n-r$ is odd. Indeed, to the authors' knowledge, in all the known roofs the bases Z_1 and Z_2 are rational homogeneous varieties and their cohomology is generated by Schubert classes.

7 Non birational Calabi–Yau pairs: the roof of type A_4^G

7.1 Duality in the space of sections

Let us recall here the geometry of the roof of typre A_4^G:

$$\begin{tikzcd}
 & & M \arrow[dl, "f_1"'] \arrow[d] \arrow[dr, "f_2"] & & \\
 & & F \arrow[dl, "p"] \arrow[dr, "q"'] & & \\
X \arrow[r, hook] & G(2, V_5) & & G(3, V_5) & Y \arrow[l, hook']
\end{tikzcd} \tag{7.1.1}$$

The notation is the following:

- V_5 is a five-dimensional vector space and $F = F(2, 3, V_5)$.
- p and q are the natural projections from F to the two Grassmannians.
- The flag variety F has Picard group generated by the pullbacks of the hyperplane bundles of the two Grassmannians $G(2, V_5)$ and $G(3, V_5)$. We denote the pullbacks of $\mathcal{O}_{G(2,V_5)}(1)$ and $\mathcal{O}_{G(3,V_5)}(1)$ by $\mathcal{O}(1, 0)$ and $\mathcal{O}(0, 1)$ respectively. In this notation M is the zero locus of a section $s \in H^0(F, \mathcal{O}(1, 1))$.

- One has that $p_*\mathcal{O}(1,1) = \mathcal{Q}_2^\vee(2)$ and $q_*\mathcal{O}(1,1) = \mathcal{U}_3(2)$, where we call $\mathcal{U}_i$ the universal bundle of a Grassmannian $G(i, V_5)$ and $\mathcal{Q}_i$ its universal quotient bundle. The varieties X and Y are, respectively, the zero loci of the sections p_*s and q_*s of $\mathcal{Q}_2^\vee(2)$ and $\mathcal{U}_3(2)$,
- f_1 is a fibration over $G(2, V_5)$ with fiber isomorphic to $\mathbb{P}^1$, for points outside the subvariety X whereas the fibers are isomorphic to $\mathbb{P}^2$ for points on X. Similarly f_2 is a map onto $G(3, V_5)$ whose fibers are $\mathbb{P}^1$ outside Y and $\mathbb{P}^2$ over Y.

The rest of this chapter is focused on proving that the general Calabi–Yau pair associated to the roof of type A_4, in the sense of Definition 4.1.8 is non birationally equivalent.

Lemma 7.1.1. *Let X be the zero locus of a regular section*

$$s_2 \in H^0(G(2, V_5), \mathcal{Q}_2^\vee(2)).$$

Then s_2 is uniquely determined by X up to scalar multiplication. Similarly, if Y is the zero locus of a regular section s_3 of $\mathcal{U}_3(2)$ on $G(3, V_5)$, s_3 is uniquely determined by Y.

Proof. We will prove the result for $G(2, V_5)$, the proof for the case of $G(3, V_5)$ is identical. Let us suppose X is the zero locus of two sections s_2 and $\widetilde{s}_2$. Then, the Koszul resolutions with respect to these two sections can be extended to the diagram:

$$\begin{tikzcd} \cdots \arrow[r] & \mathcal{Q}_2(-2) \arrow[r, "\alpha_{s_2}"] \arrow[d, "\beta"] & \mathcal{I}_X \arrow[r] \arrow[d, equal] & 0 \\ \cdots \arrow[r] & \mathcal{Q}_2(-2) \arrow[r, "\alpha_{\widetilde{s}_2}"] & \mathcal{I}_X \arrow[r] & 0 \end{tikzcd} \tag{7.1.2}$$

where the existence of the arrow β is given by the following claim:

Claim. The map

$$\mathrm{Hom}(\mathcal{Q}_2(-2), \mathcal{Q}_2(-2)) \longrightarrow \mathrm{Hom}(\mathcal{Q}_2(-2), \mathcal{I}_X) \tag{7.1.3}$$

is surjective.
This can be verified by proving surjectivity of the following map:

$$H^0(\mathcal{Q}_2^\vee \otimes \mathcal{Q}_2) \longrightarrow H^0(\mathcal{Q}_2^\vee(2) \otimes \mathcal{I}_X) \tag{7.1.4}$$

This can be achieved by tensoring the Koszul resolution of $\mathcal{I}_X$ by $\mathcal{Q}_2^\vee(2)$. In fact, by the identities $\det \mathcal{Q}_2 = \mathcal{O}(1)$ and $\mathcal{Q}_2 \simeq \wedge^2 \mathcal{Q}_2^\vee(1)$ one has the exact sequence

$$0 \longrightarrow \mathcal{Q}_2^\vee(-3) \longrightarrow \mathcal{Q}_2^\vee \otimes \mathcal{Q}_2^\vee(-1) \longrightarrow \mathcal{Q}_2^\vee \otimes \mathcal{Q}_2 \longrightarrow \mathcal{Q}(2) \otimes \mathcal{I}_X \longrightarrow 0 \tag{7.1.5}$$

where by the Borel-Weil-Bott theorem one finds:

$$\begin{gathered} H^\bullet(G(2, V_5), \mathcal{Q}_2^\vee(-3)) = 0, \\ H^\bullet(G(2, V_5), \mathcal{Q}_2^\vee \otimes \mathcal{Q}_2^\vee(-1)) = \mathbb{C}[-2], \\ H^\bullet(G(2, V_5), \mathcal{Q}_2^\vee \otimes \mathcal{Q}_2) = \mathbb{C}[0]. \end{gathered} \tag{7.1.6}$$

which proves our claim.

In particular, if two sections define the same X, then the identity of the ideal sheaf lifts to an automorphism of $\mathcal{Q}_2(-2)$. However, since $\mathrm{Ext}^\bullet(\mathcal{Q}_2, \mathcal{Q}_2) = \mathbb{C}[0]$, the only possible automorphisms of $\mathcal{Q}_2(-2)$ are scalar multiples of the identity. That implies that the sections differ by multiplication with a nonzero constant. $\square$

Corollary 7.1.2. *Let $X = Z(s_2) \subset G(2, V_5)$. Then there exists a unique hyperplane section M of F such that the fiber $p|_M^{-1}(x)$ is isomorphic to $\mathbb{P}^2$ for $x \in X$ and is isomorphic to $\mathbb{P}^1$ for $x \in G(2, V_5) \setminus X$. Similarly for $Y = Z(s_3) \subset G(3, V_5)$ there exists a unique hyperplane section M of F such that the fiber $q|_M^{-1}(x)$ is isomorphic to $\mathbb{P}^2$ for $x \in Y$ and is isomorphic to $\mathbb{P}^1$ for $x \in G(3, V_5) \setminus Y$.*

Proof. We consider only the case $X = Z(s_2) \subset G(2, 5)$ the other being completely analogous. Since F is the projectivization of a vector bundle over $G(2, V_5)$, then the pushforward p_* defines a natural isomorphism

$$H^0(F, \mathcal{O}(1, 1)) = H^0(G(2, V_5), \mathcal{Q}_2^\vee(2)).$$

Hence $s_2 = p_*(s)$ for a unique $s \in H^0(F, \mathcal{O}(1, 1))$. We define $M = Z(s)$ which satisfies the assertion by the discussion above. The uniqueness of M follows from Lemma 7.1.1. Indeed, for any hyperplane section $\tilde{M} = Z(\tilde{s})$, the fibers $p|_{\tilde{M}}^{-1}(x)$ are isomorphic to $\mathbb{P}^2$ exactly for $x \in Z(p_*\tilde{s})$, but $Z(p_*\tilde{s}) = X$ only if $p_*\tilde{s}$ is proportional to s_2 which means that $\tilde{s}$ is proportional to s and this proves uniqueness. □

Let us consider an isomorphism

$$f : G(2, V_5) \longrightarrow G(3, V_5). \tag{7.1.7}$$

Every such isomorphism is induced by a linear isomorphism $T_f : V_5 \longrightarrow V_5^\vee$ in the following way:

$$f = D \circ \phi_2 : G(2, V_5) \longrightarrow G(3, V_5). \tag{7.1.8}$$

where D is the canonical isomorphism

$$D : G(i, V_5) \longrightarrow G(5 - i, V_5^\vee) \tag{7.1.9}$$

and ϕ_i is the induced action of T_f on the Grassmannian:

$$\phi_i : G(i, V_5) \longrightarrow G(i, V_5^\vee) \tag{7.1.10}$$

Similarly, we consider dual maps $f^\vee : G(3, V_5^\vee) \longrightarrow G(2, V_5^\vee)$, expressed as $f^\vee = \phi_2^\vee \circ D^\vee$.

Note that above maps f, D, ϕ_2, ϕ_3 are restrictions of linear maps between the Plücker spaces of the corresponding Grassmannians. By abuse of notation we shall use the same name for their linear extensions. We can now introduce the following notion of duality.

Definition 7.1.3. *Given an isomorphism $f : G(2, V_5) \longrightarrow G(3, V_5)$, we say $X \subset G(2, V_5)$ is f-*dual *to $Y \subset G(2, V_5)$ if $(X, f(Y))$ is a Calabi–Yau pair associated to the roof of type A_4^G, in the sense of Definition 4.1.8.*

Let us start by defining $P = \mathbb{P}(\wedge^2 V_5) \times \mathbb{P}(\wedge^2 V_5^\vee)$, where $\wedge^2 V_5$ is identified with $\wedge^3 V_5^\vee$ by means of D. In that case F is a linear section of P (in its Segre embedding) by a codimension 25 linear space.

Remark 7.1.4. Recall that (Wey03, Proposition 3.1.9) the equations of F in P are described by the following sections $s_{x^* \otimes y} \in H^0(P, \mathcal{O}(1, 1))$

$$s_{x^* \otimes y}(\alpha, \omega) = \omega(x^*) \wedge \alpha \wedge y \tag{7.1.11}$$

for $\omega \in \Lambda^2 V_5^\vee = \Lambda^3 V_5$, $\alpha \in \Lambda^2 V_5$ and for every $x^* \otimes y \in V_5^\vee \otimes V_5$.

In other words, we have

$$s_{x^*\otimes y}(\alpha, \omega) = 0 \text{ for } ([\alpha], [\omega]) \in F(2, 3, V_5) \subset \mathbb{P}(\Lambda^2 V_5) \times \mathbb{P}(\Lambda^3 V_5).$$

This defines a 25 dimensional subspace $H^0(P, \mathcal{I}_F(1, 1)) \subset H^0(P, \mathcal{O}(1, 1))$ spanned by linearly independent sections corresponding to $x^* = e_i^*$, $y = e_j$ for $i, j \in \{1 \ldots 5\}$ and a chosen basis $\{e_i\}$ for V_5.

Now, for every f as in (7.1.7) we define the following function:

$$\begin{aligned} P &\xrightarrow{\iota_f} P \\ (x, y) &\longmapsto ((f^\vee)^{-1}(y), f(x)) \end{aligned} \tag{7.1.12}$$

which induces the following map at the level of sections:

$$\begin{aligned} H^0(P, \mathcal{O}_P(1, 1)) &\xrightarrow{\widetilde{\iota}_f} H^0(P, \mathcal{O}_P(1, 1)) \\ s &\longmapsto s \circ \iota_f \end{aligned} \tag{7.1.13}$$

Note that ι_f is a linear extension of an automorphism of the flag variety $F \subset P$. It is constructed in such a way that we have that X is defined by a section $p_*(s) \in H^0(G(2, V_5), \mathcal{Q}_2^\vee(2))$ if and only if $f(X)$ is defined by $q_*(\widetilde{\iota}_f(s)) \in H^0(G(3, V_5), \mathcal{U}_3^\vee(2))$.

Our aim is to interpret f-duality in the setting above as explicitly as possible. For that we will identify $H^0(F, \mathcal{O}(1, 1))$ with a subspace $\mathcal{H}_F$ of sections in $H^0(P, \mathcal{O}(1, 1))$ invariant under our transformations. The following lemmas will be useful in the proof of non-birationality of general Calabi–Yau pairs.

Lemma 7.1.5. *The space $H^0(P, \mathcal{O}(1,1))$ decomposes as $H^0(\mathcal{I}_{F|P}(1,1)) \oplus H^0(F, \mathcal{O}(1,1))$ and the decomposition is invariant under the action of $\widetilde{\iota}_f$ for every isomorphism $f : G(2, V_5) \to G(3, V_5)$. More precisely $\widetilde{\iota}_f H^0(\mathcal{I}_{F|P}(1,1)) = H^0(\mathcal{I}_{F|P}(1,1))$ and there exists a subspace $\mathcal{H}_F \subset H^0(P, \mathcal{O}(1,1))$ isomorphic to $H^0(F, \mathcal{O}(1,1))$ such that $\widetilde{\iota}_f(\mathcal{H}_F) = \mathcal{H}_F$.*

Proof. By the Borel–Weil–Bott theorem, one has $H^0(P, \mathcal{O}(1,1)) = V_{\omega_2} \otimes V_{\omega_3}$, which is the representation space of the product of representations of weights ω_2 and ω_3 of $GL(V_5)$. By the Littlewood-Richardson rule, this space decomposes in the following way, and the decomposition is $GL(V_5)$-invariant:

$$V_{\omega_2} \otimes V_{\omega_3} \simeq V_{\omega_2+\omega_3} \oplus V_{\omega_1+\omega_4}. \tag{7.1.14}$$

Moreover, again by the Borel–Weil–Bott theorem, one has $V_{\omega_2+\omega_3} = H^0(F, \mathcal{O}(1,1))$ from which we get a surjection:

$$H^0(P, \mathcal{O}(1,1)) \longrightarrow H^0(F, \mathcal{O}(1,1)) \tag{7.1.15}$$

from which the claim follows once we set $\mathcal{H}_F := V_{\omega_2+\omega_3}$.
Alternatively, one can proceed in the following way: it is well known that $\mathrm{Aut}(F) \simeq GL(V_5) \rtimes \mathbb{Z}/2$. Moreover, the action of $\mathrm{Aut}(F)$ on F is linear and extends to an action of $\mathrm{Aut}(F)$ on P compatible with $\widetilde{\iota}_f$. It follows that $H^0(\mathcal{I}_{F|P}(1,1))$ is invariant under $\widetilde{\iota}_f$ since it is clearly invariant under $\mathrm{Aut}(F)$. Furthermore the dual action of $\mathrm{Aut}(F)$ on $P^\vee$ preserves the dual flag variety, hence $H^0(\mathcal{I}_{F^\vee|P^\vee}(1,1))$ is invariant under the dual action of $\widetilde{\iota}_f$. We can define $\mathcal{H}_F = H^0(\mathcal{I}_{F^\vee|P^\vee}(1,1))^\perp$. The latter space is invariant under $\mathrm{Aut}(F)$, so it is also invariant under $\widetilde{\iota}_f$ and the map $\mathcal{H}_F \to H^0(F, \mathcal{O}(1,1))$ defined by restriction is an isomorphism. $\square$

Note that, by construction, the action of $\widetilde{\iota}_f$ on $H^0(F, \mathcal{O}(1,1))$ corresponds to the action $\widetilde{\iota}_f$ on $\mathcal{H}_F$. It means that we can think of $H^0(F, \mathcal{O}(1,1))$ equipped with the action induced by $\widetilde{\iota}_f$ as a subset of $H^0(P, \mathcal{O}(1,1))$ invariant under the action of $\widetilde{\iota}_f$ on $H^0(P, \mathcal{O}(1,1))$.

Remark 7.1.6. Note that, by applying the procedure of Remark 7.1.4 to describe the equations of the dual flag $F^\vee$ with respect to the dual basis of V_5, one can find explicit equations defining $\mathcal{H}_F$ in terms of matrices in $H^0(P, \mathcal{O}(1,1)) \simeq M_{10\times 10}$. In particular, in our choice of basis, Equation 7.1.11 provides explicit linear conditions on the entries of 10×10 matrices to be elements of $\mathcal{H}_F$. This will be useful in the proof of Theorem 7.2.6.

Lemma 7.1.7. *The variety X is f-dual to Y if and only if there exists a constant $\lambda \in \mathbb{C}^*$ such that sections $s_X \in \mathcal{H}_F$, $s_Y \in \mathcal{H}_F$ defining X and Y respectively satisfy $\widetilde{\iota}_f(s_Y) = \lambda s_X$.*

Proof. By definition, X is f-dual to Y if there exists a section $\hat{s} \in H^0(F, \mathcal{O}(1,1))$ such that $p_*\hat{s}$ defines X while $q_*\hat{s}$ defines $f(Y)$. By Lemma 7.1.5 there then exists a unique section $s \in \mathcal{H}_F$ such that $\hat{s} = s|_F$. Now, by definition of $\widetilde{\iota}_f$, since q_*s defines $f(Y)$ we have $p_*(\widetilde{\iota}_f)^{-1}(s)$ defines Y. Furthermore by Lemma 7.1.5 we know that $(\widetilde{\iota}_f)^{-1}(s) \in \mathcal{H}_f$. We conclude from Lemmas 7.1.1 and 7.1.5 that up to multiplication by constants $s = s_X$ and $(\widetilde{\iota}_f)^{-1}(s) = s_Y$. □

From now on, let us fix a basis of V_5 inducing a dual basis on $V_5^\vee$, and natural bases on $\wedge^2 V_5$ and $\wedge^2 V_5^\vee$ which are dual to each other.

A section $s \in H^0(P, \mathcal{O}_P(1,1))$ is represented by a 10×10 matrix S in the following way

$$s : (x, y) \longmapsto \overline{y}^T S \overline{x} \tag{7.1.16}$$

where $\overline{x}$ and $\overline{y}$ are expansions of x and y in the chosen bases of $\wedge^2 V_5$ and $\wedge^2 V_5^\vee$. Once fixed our bases, ϕ_2 is represented by a 10×10 invertible matrix M_f, which is the second exterior power of the invertible matrix associated to T_f.

We can now describe very explicitly the f-duality in terms of matrices using the following.

Lemma 7.1.8. *If S is the matrix associated to $s \in H^0(P, \mathcal{O}_P(1,1))$ then the matrix associated to $\widetilde{\iota}_f(s)$ is $M_f^{-1} S^T M_f$.*

Proof. On a pair (x, y), the map ι_f acts via $\iota_f(x, y) = ((\phi_2^\vee)^{-1}(y), \phi_2(x))$. Furthermore, in our choice of basis $\phi_2(x) = M_f \overline{x}$ and $(\phi_2^\vee)^{-1}(y) = (M_f^T)^{-1} \overline{y}$.
This yields:

$$\widetilde{\iota}_f(s)(x, y) = s \circ \iota_f(x, y) = (M_f \overline{x})^T S (M_f^T)^{-1} \overline{y} = \overline{y}^T M_f^{-1} S^T M_f \overline{x} \tag{7.1.17}$$

. □

Remark 7.1.9. In (OR17, sec. 5), it is proven that $[v] \in \mathbb{P}(\mathfrak{gl}(V))$ defines a section s_v of $\wedge^2 V(1)$, whose projection to $H^0(G(2, V_5), \wedge^2 Q_2(2))$ cuts out the threefold $X_{[v]}$. Then s_v corresponds to a 10×10 matrix S that we defined in (7.1.16). Hence, from Lemmas 7.1.7 and 7.1.8 follows that $X_{[v]}$ and $X_{[v^T]}$ are D-dual. This means that our duality relation on $\mathcal{X}_{25}$ between X and Y given by the condition of (X, Y) being a Calabi–Yau pair associated to the roof of type A_4^G is equivalent to the duality notion defined in (OR17, Section 5), extending the duality defined on $\mathcal{X}_{25}$.

7.2 Non birationality of the general pair

In this section, we prove that a general section $s \in H^0(F, \mathcal{O}(1,1))$ gives rise to two non-isomorphic Calabi–Yau threefolds $X = Z(p_*s)$ and $Y = Z(q_*s)$, this result will be stated in Theorem 7.2.6. Before proving the theorem, we will discuss some auxiliary results. In (BCP20), an argument to show that every $\widetilde{X} \subset \mathcal{X}_{25}$ is contained in just one pair of Grassmannians has been explained. Using similar ideas, we will prove an analogous result for the boundary $\bar{\mathcal{X}}_{25}$ of the family, namely that every Calabi–Yau threefold in $\bar{\mathcal{X}}_{25}$ is contained in just one Grassmannian.

Lemma 7.2.1. *Let X be a Calabi–Yau threefold described as the zero locus of a section of $\mathcal{Q}_2^\vee(2)$. Then the following equalities hold for every $t \geq 0$:*

$$H^0(G(2, V_5), \mathcal{Q}_2(-t)) = H^0(X, \mathcal{Q}_2|_X(-t)); \tag{7.2.1}$$

$$H^0(G(2, V_5), \wedge^2\mathcal{Q}_2(-t)) = H^0(X, \wedge^2\mathcal{Q}_2|_X(-t)). \tag{7.2.2}$$

In particular, $H^0(X, Q_2|_X) \simeq V_5$ *and*

$$H^0(X, Q_2|_X(-t)) = H^0(X, \wedge^2 Q_2|_X(-t)) = 0$$

for t strictly positive.

Proof. Let us consider the following short exact sequence which comes from tensoring the ideal sheaf sequence of X with Q_2:

$$0 \longrightarrow \mathcal{I}_{X/G(2,V_5)} \otimes Q_2(-t) \longrightarrow Q_2(-t) \longrightarrow Q_2|_X(-t) \longrightarrow 0 \qquad (7.2.3)$$

Given this sequence, we need to show the vanishing of the first two degrees of cohomology for $\mathcal{I}_{X/G(2,V_5)} \otimes Q_2$. To do this, we consider the sequence obtained tensoring with Q_2 the Koszul resolution of the ideal sheaf of X:

$$\begin{aligned} 0 \longrightarrow Q_2(-5-t) \xrightarrow{\theta} Q_2 \otimes Q_2^\vee(-3-t) \longrightarrow \\ \longrightarrow Q_2 \otimes Q_2(-2-t) \xrightarrow{\phi} \mathcal{I}_{X/G(2,V_5)} \otimes Q_2(-t) \longrightarrow 0 \end{aligned} \qquad (7.2.4)$$

The bundles $Q_2^\vee(-5-t)$ and $Q_2 \otimes Q_2^\vee(-3-t)$ have no cohomology in degree smaller than six: this follows from the isomorphisms

$$\begin{aligned} Q_2^\vee(-5-t) \simeq \wedge^2 Q_2^\vee \otimes (\wedge^3 Q_2^\vee)^{\otimes(4+t)} \\ Q_2 \otimes Q_2^\vee(-3-t) \simeq (\wedge^3 Q_2^\vee)^{\otimes(2+t)} \otimes \wedge^2 Q_2^\vee \otimes Q_2^\vee \end{aligned} \qquad (7.2.5)$$

and by a Borel–Weil–Bott computation. This, in turn, proves that $H^{<6}(G(2,V_5), \ker(\phi)) = 0$ in (7.2.4). Similarly, one finds that $Q_2 \otimes Q_2(-2-t)$ has no cohomology in degree smaller than four, due to $Q_2 \otimes Q_2(-2-t) \simeq (\wedge^2 Q_2^\vee)^{\otimes(2+t)}$. Therefore $H^0(G(2,V_5), \mathcal{I}_{X/G(2,V_5)} \otimes Q_2) = 0$ and $H^1(G(2,V_5), \mathcal{I}_{X/G(2,V_5)} \otimes Q_2) = 0$. This, together with (7.2.3), proves our claim (7.2.1). The second equality follows from a totally analogous computation, namely it involves the tensor product of the ideal sheaf sequence with the wedge square of Q_2. □

Lemma 7.2.2. *Let X be a Calabi–Yau threefold described as the zero locus of a section of $Q_2^\vee(2)$. Then the restriction $Q_2^\vee(2)|_X$ is slope-stable.*

Proof. Consider a subobject $\mathcal{F} \subset Q_2^\vee|_X(2)$. Then, since $G(2, V_5)$ has Picard number one, we have $c_1(\mathcal{F}) = \mathcal{O}(t)$ for some t and this leads to the injection

$$0 \longrightarrow \mathcal{O} \longrightarrow \wedge^r Q_2^\vee|_X(2r - t) \tag{7.2.6}$$

where r is the rank of $\mathcal{F}$, which can be either one or two. To have $\mathcal{F}$ as a destabilizing object for $Q_2^\vee|_X(2)$, t must satisfy the following inequality of Mumford slopes

$$\frac{t}{r} = \mu(\mathcal{F}) \geq \mu(Q_2^\vee|_X(2)) = \frac{5}{3}. \tag{7.2.7}$$

On the other hand, for the injection in (7.2.6) to exist it means that $\wedge^r Q_2^\vee|_X(2r - t)$ has global sections. Let us now consider the case $r = 1$. Then $Q_2^\vee|_X(2-t)$ has sections only for $t \leq 1$, but for such values the inequality 7.2.7 cannot be satisfied. We can prove that the same happens for $r = 2$: in fact, $\wedge^2 Q_2^\vee|_X(4-t) \simeq Q_2(3-t)$ has sections only for $t \leq 3$, but the inequality 7.2.7 cannot be fulfilled with these values of t. □

Let us suppose X is contained in two Grassmannians G_1 and G_2, where the latter is the image of the former under an isomorphism of $\mathbb{P}^9$. Since both the restrictions of the normal bundles $\mathcal{N}_i|_X = \mathcal{N}_{G_i/\mathbb{P}^9}|_X = Q_{2i}^\vee(2)|_X$ are stable with the same slope, every morphism between them must be either zero or an isomorphism. Below we furthermore prove that the isomorphism class of the normal bundle determines the Grassmannian. Combining these two facts will give us the uniqueness of the Grassmannian containing X.

Lemma 7.2.3. *Let X be a Calabi–Yau threefold described as the zero locus of a section of $Q_2^\vee(2)$. Then the following isomorphism holds:*

$$H^0(\mathbb{P}^9, \mathcal{O}(1)) \simeq H^0(X, \mathcal{O}_X(1)) \tag{7.2.8}$$

Proof. The claim follows by proving separately the following claims:

$$H^0(\mathbb{P}^9, \mathcal{O}_{\mathbb{P}^9}(1)) \simeq H^0(G(2,V_5), \mathcal{O}_{G(2,V_5)}(1)) \tag{7.2.9}$$

$$H^0(\mathbb{G}(2,V_5), \mathcal{O}_{G(2,V_5)}(1)) \simeq H^0(X, \mathcal{O}_X(1)) \tag{7.2.10}$$

Let us begin by verifying Equation 7.2.10. A twist of the Koszul resolution of $X \subset G(2,V_5)$ yields:

$$0 \longrightarrow \mathcal{O}(-4) \longrightarrow \wedge^2 Q_2(-3) \longrightarrow Q_2(-1) \longrightarrow \mathcal{O}(1) \longrightarrow i_*\mathcal{O}_X(1) \longrightarrow 0 \tag{7.2.11}$$

where i is the embedding of X in $G(2,V_5)$. The desired isomorphism is a consequence of the vanishing of cohomology of the first three bundles. To prove the validity of Equation 7.2.9 we follow basically the same argument applied to the Pfaffian resolution of $G(2,V_5)$, yielding the following exact sequence on $\mathbb{P}^9$:

$$\begin{aligned} 0 \longrightarrow \mathcal{O}(-4) \longrightarrow V_5 \otimes \mathcal{O}(-2) \longrightarrow V_5 \otimes \mathcal{O}(-1) \longrightarrow \\ \longrightarrow \mathcal{O}_{\mathbb{P}^9}(1) \longrightarrow j_*\mathcal{O}_{G(2,V_5)}(1) \longrightarrow 0 \end{aligned} \tag{7.2.12}$$

where j is the embedding of $G(2,V_5)$ in $\mathbb{P}^9$. Again the first three bundles have no cohomology. For both computations, the vanishings can be computed by the Borel–Weil–Bott theorem (or, in the first case, by Lemma 7.2.1). □

Lemma 7.2.4. *Consider a Calabi–Yau threefold $X \in \overline{X}_{25}$ such that X is contained in two translates G_1, G_2 of $G(2, V_5)$ in $\mathbb{P}^9$. If $\mathcal{N}_{G_1|\mathbb{P}(\wedge^2 V_5)}|_X \simeq \mathcal{N}_{G_2|\mathbb{P}(\wedge^2 V_5)}|_X$, then $G_1 = G_2$.*

Proof. Let $\mathcal{E}$ be a globally generated rank three vector bundle on X such that $H^0(X, \mathcal{E}) = V_5$. Then it defines a unique morphism $f : X \longrightarrow G(2, V_5)$ such that $f^*\mathcal{Q}_2 \simeq \mathcal{E}$ (Arr96, Proposition 2.1). For $i \in \{1; 2\}$, let us now apply such result to the choice $\mathcal{E} = \mathcal{N}^\vee_{X|G_i}(2) = \mathcal{Q}_2|_X$ (recall that, by Lemma 7.2.1, $H^0(G(2, V_5), \mathcal{Q}_2) \simeq H^0(X_1, \mathcal{Q}_2) \simeq V_5$). This proves that the normal bundle $\mathcal{N}_{X|G_i}$ determines the map $f_i : X \longrightarrow G_i$ up to automorphisms of G_i.
Since $\wedge^3\mathcal{Q}_2 \simeq \mathcal{O}(1)$ and by Lemma 7.2.3 one has $H^0(X, \mathcal{O}(1)) \simeq H^0(G_i, \mathcal{O}(1)) \simeq \wedge^2 V_5$, we deduce that $H^0(X, \mathcal{O}_X(1)) \simeq \wedge^3 H^0(X, \mathcal{Q}_2|_X) \simeq \wedge^2 V_5$. Then f_i, by composition with the Plücker embedding J, gives the embedding $\phi_{\mathcal{O}_X(1)} : X \hookrightarrow \mathbb{P}(\wedge^2 V_5)$ defined by $\mathcal{O}_X(1)$:

$$\begin{tikzcd} & G_i \arrow[d, hook, "J"] \\ X \arrow[ur, hook, "f_i"] \arrow[r, hook, "\phi_{\mathcal{O}_X(1)}"'] & \mathbb{P}(\wedge^2 V_5) = \mathbb{P}(H^0(X, \mathcal{O}_X(1))) \end{tikzcd} \tag{7.2.13}$$

The proof is concluded by observing that since $\mathcal{N}_{G_1|\mathbb{P}(\wedge^2 V_5)}|_X \simeq \mathcal{N}_{G_2|\mathbb{P}(\wedge^2 V_5)}|_X$, one has $f_1 = f_2$, and hence $G_1 = G_2$. □

Corollary 7.2.5. *If $X \subset \mathbb{P}^9$ is a Calabi–Yau threefold from the family*

$\bar{X}_{25}$*, then X is contained as a zero locus of a vector bundle in a unique Grassmannian* $G(2,5)$ *in its Plücker embedding.*

Proof. Suppose that X is contained in two Grassmannians G_1, G_2 such that for each of them we have an exact sequence:

$$0 \to \mathcal{N}_{X|G_i} \to \mathcal{N}_{X|\mathbb{P}^9} \to \mathcal{N}_{G_i|\mathbb{P}^9}|_X \to 0$$

Combining the two exact sequences we obtain a map: $\phi : \mathcal{N}_{X|G_1} \to \mathcal{N}_{G_2|\mathbb{P}^9}|_X$. Note that we have $\mathcal{N}_{X|G_i} \simeq \mathcal{N}_{G_i|\mathbb{P}^9}|_X \simeq Q_{2i}^\vee(2)|_X$. By stability of $Q_{2i}^\vee(2)$ we have ϕ is either trivial or an isomorphism. If it is an isomorphism it induces an isomorphism $\mathcal{N}_{G_1|\mathbb{P}^9}|_X \simeq \mathcal{N}_{G_2|\mathbb{P}^9}|_X$ and we conclude by Lemma 7.2.4. If it is trivial it lifts to an isomorphism $\mathcal{N}_{X|G_1} \simeq \mathcal{N}_{X|G_2}$ which again gives an isomorphism $\mathcal{N}_{G_1|\mathbb{P}^9}|_X \simeq \mathcal{N}_{G_2|\mathbb{P}^9}|_X$ and permits us to conclude again by Lemma 7.2.4.

□

Now we are ready to prove the main theorem of this chapter.

Theorem 7.2.6. *Let F be the partial flag manifold* $F(2,3,V_5)$*, let p and q be the projections to the two Grassmannians* $G(2,V_5)$ *and* $G(3,V_5)$*. Then a general section* $s \in H^0(F, \mathcal{O}(1,1))$ *gives rise to two non-birational Calabi–Yau threefolds* $X = Z(p_*s)$ *and* $Y = Z(q_*s)$*.*

Proof. Let (X,Y) be a Calabi–Yau pair associated to the roof of type A_4^G. Because of Lemma 7.1.1, we deduce that if there exists an isomorphism mapping X to Y, then it is given by a map $f : G(2,V_5) \to G(3,V_5)$. Recall that such a map is determined by a linear isomorphism from $T_f : V_5 \to V_5^\vee$.

Thus, because of Corollary 7.2.5, X and Y are isomorphic only if there exists $f : G(2, V_5) \to G(3, V_5)$ such that X is f-dual to X. This, by Corollary 7.1.8 translates to the fact that a section $s_X \in \mathcal{H}_F$ from Lemma 7.1.5 defining X on F satisfies $M_f^{-1} S^T M_f = \lambda S$ for S being the matrix associated to the section s_X and some constant λ. But since S and S^T are similar matrices then multiplication by λ must then preserve the spectrum of S. The proof amounts now to find a matrix S corresponding to an element of $\mathcal{H}_F$ with spectrum that is not fixed by multiplication with $\lambda \neq 1$ and such that the equation

$$S^T M - MS = 0$$

has no solutions among matrices M of the form $M = \wedge^2 T$, and then expand by openness to the general element of $\mathcal{H}_F$. This is done via the following script in Macaulay2 (GS19):

```
R=QQ[a_1..a_25]
S=matrix{
{ 1 ,0,0,0,0,0,0,0,0,0},
{0, 2 ,0,0,0,0,0,0,0,0},
{0,0, 0 ,0,0,0,0,0,0,0},
{0,0,0, 0 ,0,0,0,0,0,0},
{0,1,0,0, 0 ,0,0,0,0,0},
{0,0,0,0,0, 1 ,0,0,0,0},
{0,0,0,0,0,0,-1 ,0,0,0},
{0,0,0,0,0,0,0,-1 ,0,0},
{0,0,0,0,0,0,0,0,-1 ,0},
```

```
{0,0,0,0,0,0,0,0,0,-1 }}
T=genericMatrix(R,5,5)
M=exteriorPower(2,T)
Sol=ideal flatten(transpose(S)*M-M*S)
saturate(Sol, ideal det T)
```

Here we chose a matrix S satisfying the equations defining $\mathcal{H}_F = H^0(I_{F^\vee|P^\vee})^\perp$ as in Remark 7.1.6.

This implies that a general hyperplane section s of the flag variety F yields two Calabi–Yau threefolds X and Y which are dual, but not projectively isomorphic. By the fact that the studied manifolds have Picard number one we conclude that they are not birational (OR17, proof of Theorem 4.1). $\square$

The proof above being very explicit has the advantage that it permits to construct concrete examples of pairs of Calabi–Yau varieties in our family which are dual but not birational. We can however perform a more conceptual proof, which is more suitable to generalization and allows to estimate the expected codimension of the fixed locus of our duality.

7.3 An alternative proof

Let us present here an alternative argument for the key step of the proof of Theorem 7.2.6, based on Kleiman's transversality of a general translate. Before that, let us give the following preparatory lemma, which is a simple result of linear algebra:

Lemma 7.3.1. *Let S be a general element in $\mathcal{H}_F$. The space of matrices $M \in GL(\wedge^2 V)$ which satisfy $SM = MS^T$ is a 10 dimensional subset of symmetric matrices $\mathcal{M}_J$.*

Proof. Let us first observe that S is a matrix with distinct nonzero eigenvalues and whose spectrum is not preserved by multiplication with any $\lambda \in \mathbb{C} \setminus \{1\}$. This can be checked in a specific example and expanded by openness.

Let us fix a basis of $H^0(P, \mathcal{O}_P(1,1))$ and put S in Jordan normal form $S = J^{-1}DJ$ where D is a diagonal matrix with distinct nonzero entries. Then, $J^{-1}DJM = MJ^TDJ^{-T}$ leads to the conclusion that JMJ^T commutes with D, hence it is diagonal. It follows that M is symmetric. The dimension of $\mathcal{M}_J$ follows from the fact that once we fix J, one has

$$\mathcal{M}_J = \left\{M \in \mathrm{Sym}^2(\wedge^2 V) : \exists R \in \mathrm{diag}(10) \text{ with } M = J^{-1}RJ^{-T}\right\}. \tag{7.3.1}$$

where diag(10) denotes the subspace of $M_{10\times 10}$ of diagonal matrices. □

Theorem 7.3.2. *There exists $S \in \mathcal{H}_F$ whose spectrum is not fixed by multiplication with $\lambda \neq 1$ and such that there is no element $M \in \wedge^2 GL(V)$ satisfying the equation $SM = MS^T$.*

Proof. Define $\mathcal{T} := \{(J, M) \in GL(\wedge^2 V) \times \mathrm{Sym}^2(\wedge^2 V) : JMJ^T \text{ is diagonal }\}$ with its natural morphism q to $\mathrm{Sym}^2(\wedge^2 V)$. We introduce the notation

$\wedge^2 GL(V)$ to denote the following set:

$$\wedge^2 GL(V) := \{M \in GL(\wedge^2 V) : \exists\, T \in GL(V) \text{ satisfying } M = \wedge^2 T\}. \tag{7.3.2}$$

Consider a fibration θ fitting in the following diagram:

$$\begin{tikzcd}
 & \mathcal{T} \arrow[dl, "r"'] \arrow[dr, "q"] \arrow[ddll, bend right, "p"'] & & \\
GL(10) \arrow[d, "\theta"'] & & \mathrm{Sym}^2(\wedge^2 V) & \mathrm{Sym}^2(\wedge^2 V) \cap \wedge^2 GL(V) \arrow[l, hook'] \\
B & & &
\end{tikzcd} \tag{7.3.3}$$

Suppose the following conditions are satisfied:

1. the fibration θ has relative dimension 25 and for every $b \in B$ the fiber $\theta^{-1}(b)$ intersects with nonnegative dimension the following set:

$$\begin{aligned} \mathcal{V} = \{K \in GL(10) \mid \exists\, D \text{ diagonal with distinct} \\ \text{nonzero eigenvalues satisfying } K^{-1}DK \in \mathcal{H}_F\} \end{aligned} \tag{7.3.4}$$

2. the morphism q is flat

Then, by (Klei74, Lemma 1) and by Lemma 7.3.1, the dimension of $\mathcal{W} := p^{-1}(b) \times_{\mathrm{Sym}^2(\wedge^2 V)} (\mathrm{Sym}^2(\wedge^2 V) \cap \wedge^2 GL(V))$ can be computed in the following way:

$$\begin{aligned}\dim(\mathcal{W}) &= \dim(p^{-1}(b)) + \dim(\mathrm{Sym}^2(\wedge^2 V) \cap \wedge^2 GL(V)) + \\ &\quad - \dim(\mathrm{Sym}^2(\wedge^2 V)) \\ &= 25 + 10 + 15 - 55 = -5.\end{aligned} \tag{7.3.5}$$

where we used the fact that $\mathrm{Sym}^2(\wedge^2 V) \cap \wedge^2 GL(V)$ has dimension 15.

This shows that for every $G \in \theta^{-1}(b) \subset GL(\wedge^2 V)$ and every D diagonal with distinct nonzero eigenvalues there is no solution to $SM = MS^T$ where $S = G^{-1}DG$ and $M \in \wedge^2 GL(V)$. By our choice of fibration the latter includes some $S \in \mathcal{H}_F$, and this completes the proof.

To prove that our setting satisfies Condition 1 let us consider the following diagram:

$$\begin{tikzcd} & GL(\wedge^2 V) \times \mathbb{C}^{10} \arrow[dl, "k"'] \arrow[dr, "h"] & & \\ GL(\wedge^2 V) & & \mathcal{H}_P \arrow[r, hookleftarrow] & \mathcal{H}_F \end{tikzcd} \tag{7.3.6}$$

where $h(J, D) = J^{-1}DJ$ and k is the projection to the first factor. One has that the dimension of the general fiber $h^{-1}(S)$ is 10, hence $\dim(h^{-1}(\mathcal{H}_F)) = 75 + 10 = 85$. Observe that $\mathcal{V}$ is the image of $h^{-1}(\mathcal{H}_F)$ under k, and it has dimension 75. Let us now pick a fibration $\theta : GL(\wedge^2 V) \longrightarrow B$ of relative dimension 25. Then, given any $b \in B$, the intersection of $\theta^{-1}(b)$ with $\mathcal{V}$ has dimension 25 + 75- 100 = 0. This proves that it is possible to construct a fibration θ of relative dimension 25 such that every fiber over B intersects $\mathcal{V}$ in $GL(\wedge^2 V)$.

Let us now focus on Condition 2. First, by generic flatness, on has that there exists an open set $\mathcal{U} \subset \mathrm{Sym}(\wedge^2 V)$ such that q is flat over $\mathcal{U}$. But then one can proceed by observing that there exists a transitive $GL(10)$-action on $\mathrm{Sym}(\wedge^2 V)$ given by $G\dot{M} = GMG^T$. If q is $GL(10)$-homogeneous, then flatness of q follows from the fact that each restriction $q^{-1}(G \cdot \mathcal{U}) \longrightarrow G \cdot \mathcal{U}$ is isomorphic to $q^{-1}(\mathcal{U}) \longrightarrow \mathcal{U}$ and the translates of $\mathcal{U}$, by transitivity of the $GL(10)$-action, give an open cover of $\mathrm{Sym}(\wedge^2 V)$. We can prove that q is homogeneous by describing a $GL(10)$-action omn $\mathcal{T}$ which is compatible with q and the $GL(10)$-action on the base $\mathrm{Sym}(\wedge^2 V)$. Such action is given by the following diagram:

$$\begin{tikzcd}
(J, M) \arrow[r, mapsto, "q"] \arrow[d, mapsto, "G"] & M \arrow[d, mapsto, "G"] \\
(JG^{-1}, GMG^T) \arrow[r, mapsto, "q"] & GMG^T
\end{tikzcd} \tag{7.3.7}$$

□

Corollary 7.3.3. *If $\tilde{X}$, $\tilde{Y}$ are general Calabi–Yau threefolds in $\mathcal{X}_{25}$ which are dual in the sense of (OR17; BCP20) then they are not birational.*

Proof. By the discussion of Remark 7.1.9, the duality in $\mathcal{X}_{25}$ degenerates to $\bar{\mathcal{X}}_{25}$ to the duality relation between X and Y given by the condition of (X, Y) being a Calabi–Yau pair associated to the roof of type A_4^G. Consider an open neighborhood $\mathfrak{U} \subset \mathcal{X}_{25}$ of a general $X \in \bar{\mathcal{X}}_{25}$. Consider also the family $\mathfrak{V}$ of duals parametrized by $\mathfrak{U}$. Now

$\mathfrak{U}$ and $\mathfrak{V}$ are families of polarized Calabi–Yau threefolds such that, by Theorem 7.2.6, there exists a fiber of $\mathfrak{U}$ which is not isomorphic to the corresponding fiber of $\mathfrak{V}$. Then by the Matsusaka–Mumford theorem (MM86) the corresponding general fibers are not isomorphic and consequently general dual pairs in $\mathcal{X}_{25}$ are not isomorphic. $\square$

8 Pairs of $K3$ surfaces

8.1 Transcendental lattice and the derived global Torelli theorem

Let us recall some basic facts about lattice theory for $K3$ surfaces. For Calabi–Yau manifolds Y of dimension $n \geq 2$ there exists an isomorphism $f : \mathrm{Pic}(Y) \longrightarrow H^2(Y, \mathbb{Z})$ as a consequence of the condition $H^k(Y, \mathcal{O}) = 0$ for $1 \leq k \leq n-1$ and the exponential sequence. However, for $K3$ surfaces, f is not surjective: hence the exponential sequence yields a natural embedding of the Picard group as a submodule of $H^2(Y, \mathbb{Z})$.

Let us see this embedding from a Hodge-theoretical point of view: we call an integral Hodge structure of $K3$ type a free $\mathbb{Z}$-module V with a direct sum decomposition of its complexification

$$V \otimes \mathbb{C} = \bigoplus_{p+q=2} V^{p,q}$$

such that:

1. $V^{p,q} = \overline{V^{q,p}}$
2. $\dim V^{2,0} = 1$
3. $V^{p,q} = 0$ for $|p - q| > 2$

Moreover, we say a Hodge structure V of weight n is *polarizable* if there exists a map

$$V \otimes V \longrightarrow \mathbb{Q}(-n)$$

which fulfills the following requirements (see (Huy16, Definition 1.6) for additional details):

1. it is a morphism of Hodge structures, where we consider $\mathbb{Q}(-n)$ a Hodge structure of weight $2n$ whose only nontrivial direct summand is the one of bidegree (n, n).

2. its $\mathbb{R}$-linear extension is a bilinear form defined by a symmetric matrix N via $\alpha, \beta \longmapsto (\alpha, N\beta)$.

Clearly, the $\mathbb{Z}$-module $H^2(Y, \mathbb{Z})$ for a $K3$ surface Y can be seen as a $K3$-type polarizable Hodge structure of weight 2 where the polarization is given by the intersection product, and the embedding of the Picard group in $H^2(Y, \mathbb{Z})$ is a morphism of Hodge structures since the Picard group is given by the Hodge classes $H^{1,1}(Y) \cap H^2(Y, \mathbb{Z})$.

Observe that the intersection pairing gives $H^2(Y, \mathbb{Z})$ the structure of a lattice, we can thus identify the Picard sublattice $Pic(Y) \subset H^2(Y, \mathbb{Z})$ observing that the restriction of the intersection pairing to the Picard group is nondegenerate (Huy16, Proposition 2.4).

Another important sublattice of $H^2(Y, \mathbb{Z})$ is the *transcendental lattice T_Y* defined Hodge-theoretically as the minimal sub-Hodge structure containing $H^{2,0}(Y)$ such that $H^2(Y, \mathbb{Z})/T_Y$ is torsion-free. Moreover, the transcendental lattice of a polarizable Hodge structure of $K3$ type is itself a polarizable irreducible Hodge structure of $K3$ type, and $T_Y = \mathrm{Pic}(Y)^{\perp}$ (Huy16, Lemma 2.7).

The following theorem (Orl03, Proposition 4.2.3) provides a practical tool to understand whether two $K3$ surfaces are derived equivalent:

Theorem 8.1.1. *Let $Y, \widetilde{Y}$ be complex algebraic $K3$ surfaces. Then $D^bY \simeq D^b\widetilde{Y}$ if and only if there exists a Hodge isometry*

$$f : T_Y \longrightarrow T_{\widetilde{Y}}.$$

The following result is a consequence of Theorem 8.1.1 and the Hodge structure of a hyperplane section of a roof, as we described it in Chapter 6.

Proposition 8.1.2. *Let $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ be vector bundles of rank r on rational homogeneous bases B and $\widetilde{B}$. Let $X \simeq \mathbb{P}(\mathcal{E}^\vee) \simeq \mathbb{P}(\widetilde{\mathcal{E}}^\vee)$ be a roof of dimension $r+2$ and $(Y, \widetilde{Y})$ the Calabi–Yau pair associated to X defined by a hyperplane section $S \in H^0(X, \mathcal{L})$. Then Y and $\widetilde{Y}$ are derived equivalent.*

Proof. Given the dimension and rank of the Mukai pairs, Y and $\widetilde{Y}$ are $K3$ surfaces. By the fact that B and $\widetilde{B}$ are rational homogeneous, Equation 6.2.5 provides an isometry of transcendental lattices $T_Y \simeq T_{\widetilde{Y}}$. This, in turn, by Theorem 8.1.1, proves that Y and $\widetilde{Y}$ are derived equivalent. $\square$

8.2 Non isomorphic $K3$ pairs

Let us consider a roof X where the bases of its vector bundles are smooth quadrics Q and $\widetilde{Q}$, denote by L the line bundle which restricts to $\mathcal{O}(1)$ on each fiber of both the projective bundle structures of X. Call M the zero locus of a general section of L such that the associated Calabi–Yau pairs have dimension two (see Chapter 4 for details). Both Q and $\widetilde{Q}$ have cohomology generated by algebraic classes, hence we have an isometry $(\widetilde{j}_*\circ\widetilde{q}^*|_{T_{\widetilde{Y}}})^{-1}\circ j_*\circ q^*|_{T_Y}$ between transcendental lattices $T_Y \simeq T_{\widetilde{Y}}$ for each associated Calabi–Yau pair $(Y, \widetilde{Y})$ (in fact K3 pair in this case). Moreover, if Y and $\widetilde{Y}$ are general, by (Ogu01, proof of Lemma 4.1) T_Y and $T_{\widetilde{Y}}$ admit no self-isometries different from $\pm\,\mathrm{Id}$. Hence, to prove that Y and $\widetilde{Y}$ are not isomorphic it is enough to prove that none of the isometries $\pm(\widetilde{j}_* \circ \widetilde{q}^*|_{T_{\widetilde{Y}}})^{-1} \circ j_* \circ q^*|_{T_Y}$ extends to an isometry between $H^2(Y, \mathbb{Z})$ and $H^2(\widetilde{Y}, \mathbb{Z})$. We will prove it by studying the action of the constructed isometry on the discriminant groups of the associated transcendental lattices. For that we will consider the following notation. Given a lattice R, let us call dR the discriminant group defined by the exact sequence

$$0 \longrightarrow R \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Z}) \longrightarrow dR \longrightarrow 0. \tag{8.2.1}$$

Recall that if Y is a K3 surface of Picard number one, with L denoting the generator of its Picard group, then $\langle L\rangle = (T_Y)^{\perp} \subset H^2(Y, \mathbb{Z})$ and since $H^2(Y, \mathbb{Z})$ is unimodular $dT_Y \simeq d\langle L\rangle \simeq \mathbb{Z}/L^2\mathbb{Z}$ and there is a distinguished generator of dT_Y corresponding to $[\frac{L}{L^2}]$ under the canonical identification $dT_Y \simeq d\langle L\rangle$. Similarly, $\widetilde{Y}$ is a K3 surface of Picard number one and if we denote by $\widetilde{L}$ the generator of its Picard group, we have $[\frac{\widetilde{L}}{\widetilde{L}^2}]$ representing the generator of $dT_{\widetilde{Y}}$ associated to the embedding $T_{\widetilde{Y}} \subset H^2(\widetilde{Y}, \mathbb{Z})$.

Note that under the generality assumption for $Y, \widetilde{Y}$ (using (Ogu01, proof of Lemma 4.1)) each of the lattices T_Y, $T_{\widetilde{Y}}$ can be identified with T_M in a unique way up to $\pm\,\mathrm{Id}$, and this identification is given by $\pm j_* \circ q^*|_{T_Y}$ and $\pm \widetilde{j}_* \circ \widetilde{q}^*|_{T_{\widetilde{Y}}}$. Furthermore, $H^k(M, \mathbb{Z})$ is unimodular and hence dT_Y and $dT_{\widetilde{Y}}$ admit canonical identifications with $dH^{2r}_{alg}(M, \mathbb{Z})$. On the other hand, by Theorem 6.1.1 and Lemma 6.2.1 both $H^2(Y, \mathbb{Z})$ and $H^2(\widetilde{Y}, \mathbb{Z})$ admit Hodge isometric embeddings into $H^{2r}(M, \mathbb{Z})$ extending the embeddings of the transcendental lattices. We conclude that under our identifications $[\frac{L}{L^2}] = \pm[\frac{j_* q^* L}{L^2}] \in dH^{2r}_{alg}(M, \mathbb{Z})$ and $[\frac{\widetilde{L}}{\widetilde{L}^2}] = \pm[\frac{\widetilde{j}_* \widetilde{q}^* \widetilde{L}}{\widetilde{L}^2}] \in dH^{2r}_{alg}(M, \mathbb{Z})$. To prove that Y and $\widetilde{Y}$ are not isomorphic it remains to check that $[\frac{j_* q^* L}{L^2}]$ and $\pm[\frac{\widetilde{j}_* \widetilde{q}^* \widetilde{L}}{\widetilde{L}^2}]$ are distinct elements in $dH^{2r}_{alg}(M, \mathbb{Z})$. Indeed, if Y and $\widetilde{Y}$ were isomorphic then the isomorphism would need to map L to $\widetilde{L}$ and identify $[\frac{L}{L^2}]$ with $\pm[\frac{\widetilde{L}}{\widetilde{L}^2}]$ in $dT_M = dH^{2r}_{alg}(M, \mathbb{Z})$, contradicting the fact that that $[\frac{j_* q^* L}{L^2}] \neq \pm[\frac{\widetilde{j}_* \widetilde{q}^* \widetilde{L}}{\widetilde{L}^2}]$ in $dH^{2r}_{alg}(M, \mathbb{Z})$.

This is checked in each of the two known cases by the following Lemma.

Lemma 8.2.1. *Let X be a roof of type $G_2^\dagger$ or D_4, $M \subset X$ a general hyperplane and $Y, \widetilde{Y}$ the associated pair of K3 surfaces of degree* 12. *Then there is a unique isometry of transcendental lattices $T_Y \simeq T_{\widetilde{Y}}$ up to* $\pm\,\mathrm{Id}$ *and this isometry descends to an isomorphism of discriminant groups which maps $[\frac{L}{12}]$ to $\pm 7[\frac{\widetilde{L}}{12}]$.*

Proof. By the discussion before the Lemma we just need to compare $[\frac{j_*q^*L}{12}]$ and $[\frac{\widetilde{j}_*\widetilde{q}^*\widetilde{L}}{12}]$ in $dH^{2r}_{alg}(M,\mathbb{Z})$. For that we will present j_*q^*L and $\widetilde{j}_*\widetilde{q}^*\widetilde{L}$ in a chosen basis of $H^{2r}_{alg}(M,\mathbb{Z})$. The computations differ slightly in each of the two cases. Let us first illustrate the proof for the roof of type $G_2^\dagger$, which is easier because of the simpler structure of the cohomology ring of the quadric, which is odd dimensional.

By abuse of notation let us denote by $L \in H^2(Q,\mathbb{Z})$ the hyperplane class of Q, its restriction to Y which is the polarization, as well as its pullback to X together with its restriction to M. Fix $\xi \in H^2(X,\mathbb{Z})$ the class of the Grothendieck line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{G}^\vee(-1))}(1)$, we also denote by ξ its restriction to M.

Claim: We claim that a basis for $H^6_{alg}(M,\mathbb{Z})$ is given by the classes $\Pi, L^2\xi, L\xi^2$, where $\Pi = \frac{1}{2}L^3$ is the class of a plane in Q.

To see that, first observe that these are generators $H^6_{alg}(X,\mathbb{Z})$ which after restriction to M define a sublattice of $H^6_{alg}(M,\mathbb{Z})$.

Now, given the Grothendieck relation on X as a projective bundle on Q (Laz04a, page 310):

$$\xi^3 - 5L\xi^2 + 9L^2\xi - 12\Pi = 0 \tag{8.2.2}$$

we can write the intersection form:

	Π	$L^2\xi$	$L\xi^2$
Π	0	1	5
$L^2\xi$	1	10	32
$L\xi^2$	5	32	82

It follows that the sublattice $\langle \Pi, L^2\xi, L\xi^2\rangle \subset H^6_{alg}(M,\mathbb{Z})$ has rank 3 and discriminant 12. In light of Theorem 6.1.1 and the intersection form above, also $H^6_{alg}(M,\mathbb{Z})$ has rank 3 and discriminant 12. We conclude that $\langle \Pi, L^2\xi, L\xi^2\rangle$ is the whole $H^6_{alg}(M,\mathbb{Z})$ proving the claim.

Knowing that $(j_*q^*L)\cdot L^2\xi = (j_*q^*L)\cdot \Pi = 0$ and j_*q^*L is an effective primitive class in $H^6_{alg}(M,\mathbb{Z})$ we get:

$$j_*q^*L = L\xi^2 - 5L^2\xi + 18\Pi \tag{8.2.3}$$

which, from the relation

$$\xi = L + \widetilde{L}$$

gives

$$j_*q^*L = 7\widetilde{L}\xi^2 - 23\widetilde{L}^2\xi + 42\widetilde{\Pi}.$$

Now by the same argument repeated for $\widetilde{Y}$ we have

$$\widetilde{j}_*\widetilde{q}^*\widetilde{L} = \widetilde{L}\xi^2 - 5\widetilde{L}^2\xi + 18\widetilde{\Pi}.$$

We conclude that

$$\frac{1}{12}(j_*q^*L - 7\widetilde{j}_*\widetilde{q}^*\widetilde{L}) = \widetilde{L}^2\xi - 7\widetilde{\Pi} \in H^6_{alg}(M,\mathbb{Z}).$$

Let us now focus on the roof of type D_4. Here the $K3$ surfaces are zero loci of $\mathcal{S}^\vee(1)$. The cohomology ring of a six dimensional quadric is slightly more complicated, since there exist two disjoint families of maximal isotropic linear spaces Π_1, Π_2. They satisfy the following relations in the cohomology ring:

$$L^3 = \Pi_1 + \Pi_2,\ \ \Pi_1\cdot L = \Pi_2\cdot L,\ \ \Pi_1^2 = \Pi_2^2 = 0,\ \ \Pi_1\cdot\Pi_2 = 1. \tag{8.2.4}$$

By the same argument as above, we can construct a basis of the middle cohomology $H^8(M, \mathbb{Z})$ given by the classes $\Pi_1 L, \Pi_1\xi, \Pi_2\xi, L^2\xi^2, L\xi^3$. The Grothendieck relation is

$$\xi^4 - 6L\xi^3 + 14L^2\xi^2 - 14\Pi_1\xi - 16\Pi_2\xi + 12\Pi_1 L = 0 \tag{8.2.5}$$

which yields the following intersection matrix:

	$\Pi_1 L$	$\Pi_1\xi$	$\Pi_2\xi$	$L^2\xi^2$	$L\xi^3$
$\Pi_1 L$	0	0	0	1	6
$\Pi_1\xi$	0	0	1	6	22
$\Pi_2\xi$	0	1	0	6	22
$L^2\xi^2$	1	6	6	44	126
$L\xi^3$	6	22	22	126	308

As for the non homogeneous roof, we can compute the representation of j_*q^*L in terms of the basis above:

$$\begin{aligned} j_*q^*L &= L\xi^3 - 6L^2\xi^2 + 14\Pi_1\xi + 14\Pi_2\xi - 30\Pi_1 L \\ &= L\xi^3 - 6L^2\xi^2 + 14L^3\xi - 15L^4. \end{aligned} \tag{8.2.6}$$

where the second equality follows from the relations 8.2.4. By substituting the expression $L = \xi - \widetilde{L}$ in Equation 8.2.6 we find

$$j_*q^*L = -6\xi^4 + 29\widetilde{L}\xi^3 - 54\widetilde{L}^2\xi^2 + 46\widetilde{L}^3\xi - 15\widetilde{L}^4$$

which by Equation 8.2.5 can be rewritten as

$$j_*q^*L = -7\widetilde{L}\xi^3 + 30\widetilde{L}^2\xi^2 - 38\widetilde{\Pi}_1\xi - 50\widetilde{\Pi}_2\xi + 42\widetilde{\Pi}_1\widetilde{L}.$$

On the other hand, we can apply the same argument which leads to Equation 8.2.6 in order to get:

$$\widetilde{j}_*\widetilde{q}^*\widetilde{L} = \widetilde{L}\xi^3 - 6\widetilde{L}^2\xi^2 + 14\widetilde{\Pi}_1\xi + 14\widetilde{\Pi}_2\xi - 30\widetilde{\Pi}_1\widetilde{L}.$$

Finally, we have:

$$\frac{1}{12}(j_*q^*L + 7\widetilde{j}_*\widetilde{q}^*\widetilde{L}) = \widetilde{L}^2\xi^2 - 5\widetilde{\Pi}_1\xi - 4\widetilde{\Pi}_2\xi + 14\widetilde{\Pi}_1\widetilde{L} \in H^8_{alg}(M,\mathbb{Z}).$$

□

As a result of the discussion above, we get the following:

Corollary 8.2.2. *Let X be a roof of type $G_2^\dagger$ or D_4, $M \subset X$ a general hyperplane and $Y, \widetilde{Y}$ the associated pair of $K3$ surfaces of degree 12. Then Y and $\widetilde{Y}$ are not isomorphic.*

8.3 Fourier–Mukai transform

Let us briefly recall some standard definitions (see, for example, (Huy16)). Given a $K3$ surface S one has the following Hodge structure of weight two:

$$\widetilde{H}^{1,1}(S) = H^{1,1}(S) \oplus H^0(S) \oplus H^4(S)\ ; \quad \widetilde{H}^{2,0}(S) = H^{2,0}(S) \tag{8.3.1}$$

The Mukai lattice $\widetilde{H}(S,\mathbb{Z})$ of S is the lattice given by the Mukai pairing on $H^*(S,\mathbb{Z})$ with the Hodge structure above, where the Mukai pairing is defined by the following expression:

$$\langle\alpha,\beta\rangle = \alpha_2\cdot\beta_2 - \alpha_0\cdot\beta_4 - \alpha_4\cdot\beta_0 \tag{8.3.2}$$

Moreover, given an object $\mathcal{E} \in D^b(S)$, one defines the Mukai vector of $\mathcal{E}$ as:

$$v(\mathcal{E}) = \mathrm{ch}(\mathcal{E})\sqrt{\mathrm{td}(\mathcal{E})} = (\mathrm{rk}(\mathcal{E}), c_1(\mathcal{E}), \mathrm{ch}_2(\mathcal{E}) + \mathrm{rk}(\mathcal{E})) \tag{8.3.3}$$

Let us consider a pair $Y, \widetilde{Y}$ of $K3$ surfaces. Then, by the derived Torelli theorem, they are derived equivalent if and only if there exists a Hodge isometry of the Mukai lattice (Orl03, Theorem 4.2.1).
Let us now specialize to a general pair of $K3$ surfaces associated to a roof of type $G_2^\dagger$. Then they are derived equivalent by Theorem 8.1.2 and, by the derived Torelli theorem, it is possible to find an explicit expression of the Mukai vector of the associated cohomological Fourier–Mukai transform. This, in turn, allows us to gain some information on the Fourier–Mukai transform defining the equivalence.

Let $\widetilde{H}(Y, \mathbb{Z})$ be the Mukai lattice of Y. Then we can construct a Hodge isometry

$$\theta : \widetilde{H}(Y, \mathbb{Z}) \longrightarrow H^6_{alg}(M, \mathbb{Z}) \tag{8.3.4}$$

which can be explicitly described in terms of the basis $\{\Pi, L^2\xi, L\xi^2\}$ of $H^6_{alg}(M, \mathbb{Z})$. In particular, we must have $\theta(L) = j_*q^*L = 18\Pi - 5L^2\xi + L\xi^2$. The images $\theta(v)$ and $\theta(w)$ of the generators v of $H^0(Y, \mathbb{Z})$ and w of $H^4(Y, \mathbb{Z})$, can be determined, up to an overall sign and up to exchanging them, by the conditions $\theta(v) \cdot \theta(L) = \theta(w) \cdot \theta(L) = \theta(v) \cdot \theta(v) = \theta(w) \cdot \theta(w) = 0$ and $\theta(v) \cdot \theta(w) = 1$. Imposing such conditions we get (up to exchanging v with w or an overall sign) $\theta(v) = \Pi$ and $\theta(w) = -5\Pi + L^2\xi$. Note that, a priori, uniqueness of θ is not obvious.

Let us now consider the derived equivalence $\Phi : D^b(Y) \longrightarrow D^b(\widetilde{Y})$ given by the isometry $T_Y \simeq T_{\widetilde{Y}}$ discussed in Corollary 8.1.2. Then, by (Orl03, Theorem 4.2.1), it induces an isometry $\iota_\Phi : \widetilde{H}(Y,\mathbb{Z}) \longrightarrow \widetilde{H}(\widetilde{Y},\mathbb{Z})$, and the image $\iota_\Phi(v)$ under such isometry is the Mukai vector defining the Fourier–Mukai transform. Summing all up, we have the following commutative diagram:

$$\begin{array}{ccc} H^6_{alg}(M,\mathbb{Z}) & \xrightarrow{\text{id}} & H^6_{alg}(M,\mathbb{Z}) \\ {\scriptstyle\theta}\big\uparrow & & \big\uparrow{\scriptstyle\widetilde{\theta}} \\ \widetilde{H}(Y,\mathbb{Z}) & \xrightarrow{\iota_\Phi} & \widetilde{H}(\widetilde{Y},\mathbb{Z}) \end{array} \tag{8.3.5}$$

where the vertical arrows are given by Equation 8.3.4. Then, we can find explicitly the image $\iota_\Phi(v)$ of the generator v of $H^0(Y,\mathbb{Z})$ using $\iota_\Phi(v) = \widetilde{\theta}^{-1}(\theta(v))$. By direct computation we find the Mukai vector $(2,1,-3)$.

Proposition 8.3.1. *Let Y, $\widetilde{Y}$ be a pair of $K3$ surfaces of Picard number 1 defined by a hyperplane section M of a roof of type $G_2^\dagger$. Then $\widetilde{Y}$ is isomorphic to the moduli space $\mathcal{M}_Y(u)$ of vector bundles $\mathcal{F}$ on Y with Mukai vector*

$$v(\mathcal{F}) = u = (2,1,-3).$$

Proof. The proof follows from the computation above and the proof of (Orl03, Theorem 4.2.3). Indeed, in the reference the author identifies by means of the Torelli theorem the variety $\widetilde{Y}$ with the Moduli space $\mathcal{M}_Y(u)$ (introduced and studied in (Muk87)) of stable sheaves on Y with given Mukai vector u . □

We thus recover the well-known Fourier–Mukai transform yielding Mukai duality for $K3$ surfaces of degree 12 (Muk98, Example 1.3). This also gives an alternative proof of non-isomorphicity of Y, $\widetilde{Y}$.

Remark 8.3.2. It is tempting to extend this approach to the roof of type D_4. However, instead of the isometries θ and $\widetilde{\theta}$, one can construct isometries of $H^8_{alg}(M,\mathbb{Z})$ with a lattice of rank 5 containing a hyperbolic lattice and the Picard lattice. This construction is highly non unique, and it is not known, a priori, if a diagram such as 8.3.5 exists.

9 Derived equivalence of Calabi–Yau pairs

9.1 Setup and general strategy

Let us recall some terminology. In the following, G/P is a homogeneous roof of rank r, i.e. a homogeneous Fano variety with two different projective bundle structures $\mathbb{P}(\mathcal{E}_i) \longrightarrow G/P_i$, where G/P_i is the generalized Grassmannian given by the quotient of the reductive linear group G by the parabolic subgroup P_i, for $i \in \{1; 2\}$ (see Definition 4.1.5). Let L be the very ample line bundle which restricts to $\mathcal{O}(1)$ on each fiber of both projectivizations, whose existence follows from Proposition 4.1.4, and call $M \subset G/P$ the (smooth) zero locus of a general section of L. We will commit the abuse of notation of denoting by L alos the pullback of such line bundle to M. Let (Y_1, Y_2) be the associated Calabi–Yau pair, i.e. Y_1 and Y_2 are zero loci of pushforwards of a section defining M along the projective bundle maps (see Definition 4.1.8). One has the following diagram (Diagram 5.2.1):

$$
\begin{tikzcd}
 & T_1 \arrow[rr, hook, "k_1"] \arrow[ddl, "\nu_1"'] & & M \arrow[d, hook, "l"] \arrow[ddll, "h_1|_M"'] \arrow[ddrr, "h_2|_M"] & & T_2 \arrow[ll, hook', "k_2"'] \arrow[ddr, "\nu_2"] & \\
 & & & G/P \arrow[dll, "h_1"] \arrow[drr, "h_2"'] & & & \\
Y_1 \arrow[r, hook, "t_1"] & G/P_1 & & & & G/P_2 & Y_2 \arrow[l, hook', "t_2"']
\end{tikzcd}
\tag{9.1.1}
$$

where T_i is the preimage of Y_i under $h_i|_M$, and ν_i is the restriction of $h_i|_M$ to T_i for $i \in \{1; 2\}$.

By an application of the Cayley trick (Orl03, Proposition 2.10), one finds the following semiorthogonal decompositions:

$$\begin{aligned} D^b(M) &\simeq \langle k_{1*}\nu_1^* D^b(Y_1), h_1|_M^* D^b(G/P_1) \otimes L, \ldots, h_1|_M^* D^b(G/P_1) \otimes L^{\otimes(r-1)} \rangle \\ &\simeq \langle k_{2*}\nu_2^* D^b(Y_2), h_2|_M^* D^b(G/P_2) \otimes L, \ldots, h_2|_M^* D^b(G/P_2) \otimes L^{\otimes(r-1)} \rangle \end{aligned} \tag{9.1.2}$$

Remark 9.1.1. Note that in the case of roofs of type $A_n \times A_n$ we can proceed observing that the zero locus $M \subset \mathbb{P}^n \times \mathbb{P}^n$ of a section of L is isomorphic to a flag variety $F(1, n, n+1)$. Hence, by Orlov's formula for semiorthogonal decompositions of projective bundles (Theorem 3.3.1), we recover the same decomposition of Equation 9.1.2 except for the fact that $D^b(Y_1)$ and $D^b(Y_2)$ do not appear. This is of course not a surprise, since for roofs of type $A_n \times A_n$ the zero loci Y_1 and Y_2 are empty.

Assume that both $D^b(G/P_1)$ and $D^b(G/P_2)$ admit a known full exceptional collection of homogeneous vector bundles (see Remark 10.1.2 for the list of roofs where this is true). In this section, for several examples of roofs, we will describe a sequence of mutations realizing the following equivalence:

$$\begin{aligned} D^b(M) &\simeq \langle k_{1*}\nu_1^* D^b(Y_1), h_1|_M^* D^b(G/P_1) \otimes L, \ldots, h_1|_M^* D^b(G/P_1) \otimes L^{\otimes(r-1)} \rangle \\ &\to \langle \psi D^b(Y_1), h_2|_M^* D^b(G/P_2) \otimes L, \ldots, h_2|_M^* D^b(G/P_2) \otimes L^{\otimes(r-1)} \rangle \end{aligned} \tag{9.1.3}$$

where ψ is an equivalence functor defined by the action of mutations on the Calabi–Yau subcategory. We compare this last decomposition with the one obtained applying (Orl03, Proposition 2.10) to the right-hand side of Diagram 9.1.1:

$$D^b(M) \simeq \langle k_{2*}\nu_2^* D^b(Y_2), h_2|_M^* D^b(G/P_2) \otimes L, \dots, h_2|_M^* D^b(G/P_2) \otimes L^{\otimes(r-1)} \rangle \tag{9.1.4}$$

hence realizing an equivalence $D^b(Y_1) \simeq D^b(Y_2)$.

Note that, by construction, all exceptional objects appearing in the above semiorthogonal decomposition are pullbacks from G/P. All mutations computed in this chapter involve only objects of this kind. We will prove that such mutations fulfill also the following condition, which will be needed for further applications in Chapters 10 and 11.

Definition 9.1.2. *Consider a mutation* $(F_1, F_2) \longrightarrow (\mathbb{L}_{F_1} F_2, F_1)$ *of exceptional objects on* M*. We say that such mutation satisfies* Condition (†) *if there exist exceptional objects* E_1*,* E_2 *on* G/P *such that for* $i \in \{1; 2\}$ *one has* $l^* E_i = F_i$ *and the following vanishings hold:*

$$\mathrm{Ext}^\bullet_{G/P}(E_2, E_1) = \mathrm{Ext}^\bullet_{G/P}(E_1 \otimes L, E_2) = 0. \tag{†}$$

The same condition is defined for right mutations $(F_1, F_2) \longrightarrow (F_2, \mathbb{R}_{F_2} F_1)$.

Definition 9.1.2 will be generalized in Chapter 10, using the notion of *L-semiorthogonality* (Definition 10.3.2) which is related by Proposition 10.3.4 to the concept of mutations commuting with l^*, in the sense of Definition 10.3.1. However, at this stage, all mutations are computed directly, hence there is no need to introduce such definition here.

9.2 Derived equivalence for the roof of type C_2

Let V_4 be a vector space of dimension four. A roof of type C_2 is given by the following diagram:

$$\begin{tikzcd} & IF(1,2,V_4) \arrow[dl, "h_1"'] \arrow[dr, "h_2"] & \\ IG(1,V_4) & & IG(2,V_4) \end{tikzcd} \tag{9.2.1}$$

where IG and IF denote, respectively symplectic Grassmannians and flag varieties. Note that $IG(1,V_4) \simeq \mathbb{P}^3$ and $IG(2,V_4)$ is a three dimensional quadric in $\mathbb{P}^4$. Both h_1 and h_2 are $\mathbb{P}^1$-fibrations. Let us fix $L = \mathcal{O}(1,1) := h_1^*\mathcal{O}(1) \otimes h_2^*\mathcal{O}(1)$, choose a general section $\sigma \in H^0(IF(1,2,V_4), \mathcal{O}(1,1))$ and call $M = Z(\sigma)$ its zero locus, embedded by l in $F(1,2,V_4)$. Then, by dimensional reasons and Lemma 4.1.6, the zero loci $Y_1 = Z(h_{1*}\sigma)$ and $Y_2 = Z(h_{2*}\sigma)$ are elliptic curves.

Let us call $\mathcal{U}$ the pullback to $IF(1,2,V_4)$ of the tautological vector bundle of $IG(2,V_4)$. By Cayley trick (Orl03, Proposition 2.10) we write the following semiorthogonal decompositions:

$$\begin{aligned} D^b(M) \simeq & \langle \phi_1 D^b(Y_1), \mathcal{O}_M(-1,1), \mathcal{O}_M(0,1), \mathcal{O}_M(1,1), \mathcal{O}_M(2,1)\rangle \\ \simeq & \langle \phi_2 D^b(Y_2), \mathcal{O}_M(1,1), l^*\mathcal{U}^\vee(1,1), \mathcal{O}_M(1,2), \mathcal{O}_M(1,3)\rangle \end{aligned} \tag{9.2.2}$$

where $\phi_i = k_{i*}v_i^*$ in the notation of Diagram 9.1.1. We formulate the following lemma:

Lemma 9.2.1. *In the setting above, there is a sequence of mutations realizing a derived equivalence $D^b(Y_1) \longrightarrow D^b(Y_2)$ and satisfying Condition †.*

Proof. First, let us apply the Serre functor to the last four objects of each collection, obtaining:

$$\begin{aligned} D^b(M) \simeq&\langle \mathcal{O}_M(-2,0), \mathcal{O}_M(-1,0), \mathcal{O}_M, \mathcal{O}_M(1,0), \phi_1 D^b(Y_1)\rangle \\ \simeq&\langle \mathcal{O}_M, l^*\mathcal{U}^\vee, \mathcal{O}_M(0,1), \mathcal{O}_M(0,2), \phi_2 D^b(Y_2)\rangle \end{aligned} \tag{9.2.3}$$

For brevity, let us fix $F := IF(1,2,V_4)$. Our approach for finding the right mutations follows (Mor19) closely. Let us start from the first collection. We can send the first bundle to the far right, then move $\phi_1 D^b(Y_1)$ one step to the right, obtaining

$$D^b(M) \simeq\langle \mathcal{O}_M(-1,0), \mathcal{O}_M, \mathcal{O}_M(1,0), \mathcal{O}_M(-1,1), \mathbb{R}_{\mathcal{O}_M(-1,1)}\phi_1 D^b(Y_1)\rangle \tag{9.2.4}$$

We have the following short exact sequence on F (and its pullback on M) (Mor19, Equation 2.2):

$$0 \longrightarrow \mathcal{O}(-1,1) \longrightarrow \mathcal{U}^\vee \longrightarrow \mathcal{O}(1,0) \longrightarrow 0 \tag{9.2.5}$$

All cohomology in the following is computed by Borel–Weil–Bott's theorem. First, since one has

$$\mathrm{Ext}^\bullet_F(\mathcal{O}(1,0), \mathcal{O}(-1,1)) = \mathrm{Ext}^\bullet_M(\mathcal{O}_M(1,0), \mathcal{O}_M(-1,1)) = \mathbb{C}[-1] \tag{9.2.6}$$

we can mutate $\mathcal{O}_M(1,0)$ and get:

$$D^b(M) \simeq\langle \mathcal{O}_M(-1,0), \mathcal{O}_M, \mathcal{O}_M(-1,1), l^*\mathcal{U}^\vee, \mathbb{R}_{\mathcal{O}_M(-1,1)}\phi_1 D^b(Y_1)\rangle \tag{9.2.7}$$

and we compute the following vanishings:

$$\operatorname{Ext}^\bullet_F(\mathcal{O}(2,1),\mathcal{O}(-1,1)) = \operatorname{Ext}^\bullet_F(\mathcal{O}(-1,1),\mathcal{O}(1,0)) = 0 \tag{9.2.8}$$

which are required to fulfill Condition (†). The next step is to exchange the second and the third bundles. We have:

$$\operatorname{Ext}^\bullet_F(\mathcal{O},\mathcal{O}(-1,1)) = \operatorname{Ext}^\bullet_M(\mathcal{O}_M,\mathcal{O}_M(-1,1)) = 0 \tag{9.2.9}$$

hence we can move the first two to the end and send $\mathbb{R}_{\mathcal{O}_M(-1,1)}\phi_1 D^b(Y_1)$ to the far right. Again, this mutation fulfills Condition (†) since:

$$\operatorname{Ext}^\bullet_F(\mathcal{O}(-1,1),\mathcal{O}) = \operatorname{Ext}^\bullet_F(\mathcal{O}(1,1),\mathcal{O}(-1,1)) = 0. \tag{9.2.10}$$

We find:

$$\begin{aligned} D^b(M) \simeq \langle &\mathcal{O}_M, l^*\mathcal{U}^\vee, \mathcal{O}_M(0,1), \mathcal{O}_M(0,2), \\ &\mathbb{R}_{\mathcal{O}_M(0,2)}\mathbb{R}_{\mathcal{O}_M(0,1)}\mathbb{R}_{\mathcal{O}_M(-1,1)}\phi_1 D^b(Y_1)\rangle \end{aligned} \tag{9.2.11}$$

In the first four bundles we recognise $D^b(IG(2,V_4))$. Hence, comparing Equation 9.2.3 with Equation 9.2.11 we prove our claim. □

Remark 9.2.2. Note that the derived equivalence $D^b(Y_1) \simeq D^b(Y_2)$ is a consequence of the derived equivalence of local Calabi–Yau fivefolds described in (Mor19): in fact, one can follow the approach of (Ued19) based on matrix factorization categories. In general, given a roof of type G/P with $\mathbb{P}^{r-1}$-bundle structures $h_i : G/P \longrightarrow G/P_i$, let us call $\mathcal{E}_i := h_{i*}\mathcal{O}(1,1)$ and $Y_i = Z(h_{i*}\sigma)$, where σ is a general section of $\mathcal{O}(1,1)$. Then, one can define by the data of a section of $\mathcal{E}_i$ a superpotential w_i such that the derived category of matrix factorizations of the Landau–Ginzburg model $(\mathcal{E}_i^\vee, w_i)$ is equivalent to $D^b(Y_i)$ via Knörrer periodicity (for more details, see Chapter 14). Then, by (Ued19) if there is a derived equivalence $D^b(\mathrm{Tot}\,\mathcal{E}_1^\vee) \simeq D^b(\mathrm{Tot}\,\mathcal{E}_2^\vee)$ satisfying a $\mathbb{C}^*$-equivariancy condition, it lifts to a derived equivalence of the matrix factorization categories of $(\mathcal{E}_i^\vee, w_i)$, and $D^b(Y_1) \simeq D^b(Y_2)$ follows from this last equivalence composed with Knörrer periodicity. This gives a derived equivalence for Calabi–Yau pairs of type A_4^G, C_2 (Mor19) and G_2 (Ued19).

9.3 Derived equivalence for the roof of type A_n^M

Let V be a vector space of dimension $n+1$. A roof of type A_n^M is given by the following diagram:

$$\begin{tikzcd} & F(1,n,V) \arrow[dl, "h_1"'] \arrow[dr, "h_2"] & \\ G(1,V) & & G(n,V) \end{tikzcd} \tag{9.3.1}$$

Fix $L = \mathcal{O}(1,1) := h_1^*\mathcal{O}(1) \otimes h_2^*\mathcal{O}(1)$ and choose σ to be a general section of such bundle. Call $M = Z(\sigma)$ and define the immersion $l : M = Z(\sigma) \hookrightarrow F(1,n,V)$. Then the zero loci $Y_i = Z(h_{i*}\sigma)$ are zero-dimensional. Nonetheless we discuss their derived equivalence via mutations, since it will be necessary to prove further results in Chapters 10 and 11. A similar result has been previously found in the context of Mukai flops by (Kaw02; Nam03), later (Mor19) developed a sequence of mutations to achieve such result. The proof we will describe is similar to the one by (Mor19).

By Cayley trick we recover the following semiorthogonal decompositions:

$$\begin{aligned} D^b(M) \simeq& \langle \phi_1 D^b(Y_1), h_1|_M^* D^b G(1,V) \otimes L, \ldots, h_1|_M^* D^b G(1,V) \otimes L^{\otimes(n-1)} \rangle \\ \simeq& \langle \phi_2 D^b(Y_2), h_2|_M^* D^b G(n,V) \otimes L, \ldots, h_2|_M^* D^b G(n,V) \otimes L^{\otimes(n-1)} \rangle \end{aligned} \tag{9.3.2}$$

where $\phi_i := l_* \nu_i^*$. By choosing the right twists of Beilinson's full exceptional collection for $\mathbb{P}^n$ (Bei78) we write:

$$\begin{aligned} D^b(M) \simeq \langle \phi_1 D^b(Y_1), & \mathcal{O}_M(1,1), \ldots\ldots\ldots\ldots\ldots, \mathcal{O}_M(n+1,1), \\ & \vdots \qquad\qquad\qquad \vdots \\ & \mathcal{O}_M(n-1,n-1), \ldots, \mathcal{O}_M(2n-1,n-1)\rangle \\ \simeq \langle \phi_2 D^b(Y_2), & \mathcal{O}_M(1,1-n), \ldots\ldots\ldots\ldots\ldots, \mathcal{O}_M(1,1), \\ & \vdots \qquad\qquad\qquad \vdots \\ & \mathcal{O}_M(n-1,-1), \ldots, \ldots \mathcal{O}_M(n-1,n-1)\rangle \end{aligned} \tag{9.3.3}$$

First, we need the following vanishing results:

Lemma 9.3.1. *For* $1 < m < n+1$ *one has:*

$$\mathrm{Ext}^\bullet_M(\mathcal{O}_M(m+1,1), \mathcal{O}_M(2,2)) = \mathrm{Ext}^\bullet_{F(1,n,V)}(\mathcal{O}(m+1,1), \mathcal{O}(2,2)) = 0. \tag{9.3.4}$$

Proof. We need to compute the cohomology of $\mathcal{O}(1-m,1)$ and $\mathcal{O}_M(1-m,1)$. Twisting the Koszul resolution for M yields:

$$0 \longrightarrow \mathcal{O}(-m,0) \longrightarrow \mathcal{O}(1-m,1) \longrightarrow \mathcal{O}_M(1-m,1) \longrightarrow 0 \tag{9.3.5}$$

Observe that $O(a,b)$ is flat over $G(n,V)$ (Har77, Proposition III.9.2), and for every l one has that $\dim H^l(h_2^{-1}(x), O(a)|_{h_2^{-1}(x)})$ does not depend on x. Hence, by (Mum12, Page 50, Corollary 2) for every $x \in G(n,V)$:

$$R^\bullet h_{2*}O(a,b)_x \simeq H^\bullet(h_2^{-1}(x), O(a)|_{h_2^{-1}(x)}) \tag{9.3.6}$$

This is identically zero for $a = -m$ or $a = 1-m$. Thus, by the Leray spectral sequence, both $O_M(1-m,1)$ and $O(1-m,1)$ are acyclic. □

Lemma 9.3.2. *For* $1 < m < n+1$ *the following holds:*

$$\mathrm{Ext}^\bullet_{F(1,n,V)}(O(2,2), O(m+1,1)) = \mathrm{Ext}^\bullet_{F(1,n,V)}(O(m+2,2), O(2,2)) = 0. \tag{9.3.7}$$

Proof. This is a consequence of the semiorthogonality of

$$\begin{aligned} D^b(F(1,n,V)) \simeq \langle & O_M(0,0), \ldots\ldots\ldots\ldots\ldots, O_M(n,0), \\ & \qquad\vdots \qquad\qquad\qquad\qquad \vdots \\ & O_M(n-1,n-1), \ldots, O_M(2n-1,n-1)\rangle. \end{aligned} \tag{9.3.8}$$

Alternatively, one can formulate a direct computation as in the proof of Lemma 9.3.1. □

Lemma 9.3.3. *In the setting above, there is a sequence of mutations of exceptional objects of* $D^b(M)$ *satisfying Condition (†) and realizing a derived equivalence* $D^b(Y_1) \simeq D^b(Y_2)$.

Proof. Let us switch to a more compact notation: hereafter $O_{a,b} :=$

$\mathcal{O}_M(a,b)$. Hence, Equation 9.3.2 becomes:

$$\begin{aligned} D^b(M) \simeq \langle \phi_1 D^b(Y_1), &\mathcal{O}_{1,1}, \dots\dots\dots\dots, \mathcal{O}_{n+1,1}, \\ &\mathcal{O}_{2,2}, \dots\dots\dots\dots, \mathcal{O}_{n+2,2}, \\ &\ \vdots \qquad\qquad\qquad \vdots \\ &\mathcal{O}_{n-1,n-1}, \dots\dots\dots, \mathcal{O}_{2n-1,n-1} \rangle. \end{aligned} \tag{9.3.9}$$

First, let us move $\phi_1 D^b(Y_1)$ one step to the right, then let us send $\mathcal{O}_{1,1}$ to the end of the collection. We get:

$$\begin{aligned} D^b(M) \simeq \langle \psi_1 \phi_1 D^b(Y_1), &\mathcal{O}_{2,1} \dots\dots\dots\dots\dots, \mathcal{O}_{n+1,1}, \mathcal{O}_{2,2} \\ &\mathcal{O}_{3,2}, \dots\dots\dots\dots\dots, \mathcal{O}_{n+2,2}, \mathcal{O}_{3,3} \\ &\ \vdots \qquad\qquad\qquad\qquad \vdots \\ &\mathcal{O}_{n,n-1}, \dots\dots\dots\dots, \mathcal{O}_{2n-1,n-1}, \mathcal{O}_{n,n} \rangle \end{aligned} \tag{9.3.10}$$

where $\psi_1 := \mathbb{R}_{\mathcal{O}_{1,1}}$. By Lemma 9.3.1 we can move $\mathcal{O}_{2,2}$ leftwards until it stops at the right of $\mathcal{O}_{2,1}$, since it is orthogonal to all the bundles in between. These mutations, by Lemma 9.3.2, satisfy Condition (†). We can repeat the same step on each row obtaining:

$$\begin{aligned} D^b(M) \simeq \langle \psi_1 \phi_1 D^b(Y_1), &\mathcal{O}_{2,1}, \mathcal{O}_{2,2}, \mathcal{O}_{3,1}, \dots\dots\dots\dots\dots, \mathcal{O}_{n+1,1}, \\ &\mathcal{O}_{3,2}, \mathcal{O}_{3,3}, \mathcal{O}_{4,2}, \dots\dots\dots\dots\dots, \mathcal{O}_{n+2,2}, \\ &\ \vdots \qquad\qquad\qquad\qquad\qquad \vdots \\ &\mathcal{O}_{n,n-1}, \mathcal{O}_{n+1,n-1}, \mathcal{O}_{n+2,n-1}, \dots\dots, \mathcal{O}_{2n-1,n-1} \rangle. \end{aligned} \tag{9.3.11}$$

Now we mutate $\psi_1 \phi_1 D^b(Y_1)$ two steps to the right, then we move the first two bundles to the end of the collection obtaining:

$$\begin{aligned} D^b(M) \simeq \langle \psi_2\phi_1 D^b(Y_1) \mathcal{O}_{3,1}, \dots\dots\dots\dots\dots\dots, &\mathcal{O}_{n+1,1}, \mathcal{O}_{3,2}, \mathcal{O}_{3,3}, \\ \mathcal{O}_{4,2}, \dots\dots\dots\dots\dots\dots, &\mathcal{O}_{n+2,2}, \mathcal{O}_{4,3}, \mathcal{O}_{4,4}, \\ \vdots \qquad\qquad\qquad\qquad & \quad \vdots \\ \mathcal{O}_{n+2,n-1}, \dots\dots\dots, \mathcal{O}_{2n-1,n-1}, &\mathcal{O}_{n+1,n}, \mathcal{O}_{n+1,n+1} \rangle \end{aligned} \tag{9.3.12}$$

where $\psi_2 = \mathbb{R}_{\langle \mathcal{O}_{2,1}, \mathcal{O}_{2,2} \rangle} \circ \psi_1$. Then, on each row, using Lemma 9.3.1 we shift the last two bundles all the way left, until they stop at the right of the first bundle. Again, by Lemma 9.3.2 such mutations satisfy Condition (†). We find:

$$\begin{aligned} D^b(M) \simeq \langle \psi_2\phi_1 D^b(Y_1), &\mathcal{O}_{3,1}, \mathcal{O}_{3,2}, \mathcal{O}_{3,3}, \mathcal{O}_{4,1}, \dots\dots\dots\dots\dots, \mathcal{O}_{n+1,1}, \\ &\mathcal{O}_{4,2}, \mathcal{O}_{4,3}, \mathcal{O}_{4,4}, \mathcal{O}_{5,2}, \dots\dots\dots\dots\dots, \mathcal{O}_{n+2,2}, \\ &\quad \vdots \qquad\qquad\qquad\qquad\qquad\qquad \vdots \\ &\mathcal{O}_{n+1,n-1}, \mathcal{O}_{n+1,n}, \mathcal{O}_{n+1,n+1}, \mathcal{O}_{n+2,n-1}, \dots, \mathcal{O}_{2n-1,n-1} \rangle. \end{aligned} \tag{9.3.13}$$

This process can be iterated moving the first three bundles to the end, then on each row sending the last three bundles to the right of the first one, and repeating these steps increasing by one the number of bundles we move. We stop once we get a semiorthogonal decomposition given by $n-1$ twists of $\langle \mathcal{O}_{1,1}, \dots \mathcal{O}_{1,n+1} \rangle$ and the image of $\phi_1 D^b(Y_1)$ under a composition of mutations. This eventually happens after n steps. We get the following collection:

$$
\begin{aligned}
D^b(M) \simeq \langle \psi_n\phi_1 D^b(Y_1), & \mathcal{O}_{n+1,1}, \mathcal{O}_{n+1,2}, \ldots\ldots\ldots\ldots, \mathcal{O}_{n+1,n+1}, \\
& \mathcal{O}_{n+2,2}, \mathcal{O}_{n+2,3}, \ldots\ldots\ldots\ldots, \mathcal{O}_{n+2,n+2}, \\
& \quad \vdots \qquad\qquad\qquad\qquad \vdots \\
& \mathcal{O}_{2n-1,n-1}, \mathcal{O}_{2n-1,n}, \ldots\ldots, \mathcal{O}_{2n-1,2n-1} \rangle
\end{aligned}
\tag{9.3.14}
$$

If we twist the whole collection by $\mathcal{O}_{-n,-n}$ we obtain:

$$
\begin{aligned}
D^b(M) \simeq \langle \mathcal{T}_{-n,-n} \circ \psi_n\phi_1 D^b(Y_1), & \mathcal{O}_{1,1-n}, \mathcal{O}_{1,2-n}, \ldots\ldots\ldots\ldots\ldots, \mathcal{O}_{1,1}, \\
& \mathcal{O}_{2,2-n}, \mathcal{O}_{2,3-n}, \ldots\ldots\ldots\ldots\ldots, \mathcal{O}_{2,2}, \\
& \quad \vdots \qquad\qquad\qquad\qquad \vdots \\
& \mathcal{O}_{n-1,-1}, \mathcal{O}_{n-1,0}, \ldots\ldots\ldots, \mathcal{O}_{n-1,n-1} \rangle
\end{aligned}
\tag{9.3.15}
$$

where $\mathcal{T}_{-n,-n}$ is the twist functor. The proof is concluded by comparing Equation 9.3.3 with Equation 9.3.15. □

9.4 Derived equivalence for the roof of type $A_n \times A_n$

Let us fix a vector space V_{n+1} of dimension $n+1$. The relevant diagram is:

$$
\begin{tikzcd}
& \mathbb{P}^n \times \mathbb{P}^n \arrow[dl, "h_1"'] \arrow[dr, "h_2"] & \\
\mathbb{P}^n & & \mathbb{P}^n
\end{tikzcd}
\tag{9.4.1}
$$

Here the zero locus M of a general section s of $L = \mathcal{O}(1,1) = h_1^*\mathcal{O}(1) \otimes h_2^*\mathcal{O}(1)$ is isomorphic to $F(1,n,V_{n+1})$. In the similar context of standard flops, a proof of derived equivalence by mutations has been found by (BO95) and later by (ADM19; Mor19). We will present a similar sequence of mutations adapted to the present setting.

Instead of using the Cayley trick as in the previous sections, we construct two semiorthogonal decompositions for M using the fact that it admits two projective bundle structures (M is itself a roof!) obtaining:

$$\begin{aligned} D^b(M) = \langle \mathcal{O}(0,0) & \ldots\ldots\ldots \mathcal{O}(n,0) \\ \vdots \qquad & \qquad\quad \vdots \\ \mathcal{O}(n,n) & \ldots\ldots\ldots \mathcal{O}(2n,n)\rangle \end{aligned} \tag{9.4.2}$$

$$\begin{aligned} D^b(M) = \langle \mathcal{O}(0,-n) & \ldots\ldots\ldots \mathcal{O}(0,0) \\ \vdots \qquad & \qquad\quad \vdots \\ \mathcal{O}(n,0) & \ldots\ldots\ldots \mathcal{O}(n,n)\rangle \end{aligned} \tag{9.4.3}$$

As we observed in Remark 9.1.1, there is no Calabi–Yau pair associated to this roof since the zero loci of $h_{1*}s$ and $h_{2*}s$ are empty, which makes the problem of derived equivalence somewhat trivial. Nonetheless, the existence of a sequence of mutations transforming the collection 9.4.2 into 9.4.3 will be useful for further applications in Chapter 11, therefore we formulate the following result:

Lemma 9.4.1. *In the setting introduced above, there is a sequence of mutations transforming the collection 9.4.2 into 9.4.3, and each of such mutations satisfies Condition (†).*

Proof. The approach is nearly identical to the one we used to prove Lemma 9.3.3, which is not a surprise since the collections are very similar. In fact, 9.3.3 can be formally obtained by removing one row from 9.4.2 and adding the Calabi–Yau subcategory. Hence, up to some careful adaptation of the computation of Exts, we can apply essentially the same argument. We first need to prove the following vanishings for $2 \leq k \leq n$:

$$\begin{aligned} \mathrm{Ext}^\bullet_M(\mathcal{O}(k,0), \mathcal{O}(1,1)) &= 0 \\ \mathrm{Ext}^\bullet_{\mathbb{P}^n \times \mathbb{P}^n}(\mathcal{O}(k+1,1), \mathcal{O}(1,1)) &= 0 \\ \mathrm{Ext}^\bullet_{\mathbb{P}^n \times \mathbb{P}^n}(\mathcal{O}(1,1), \mathcal{O}(k,0)) &= 0. \end{aligned} \tag{9.4.4}$$

The first vanishing has already been computed in the proof of 9.3.3. The second one follows from the fact that $\mathrm{Ext}^\bullet_{\mathbb{P}^n \times \mathbb{P}^n}(\mathcal{O}(k+1,1), \mathcal{O}(1,1)) = H^\bullet(\mathbb{P}^n \times \mathbb{P}^n, h_1^*\mathcal{O}(-k))$ which is zero, while the third one can be proved in the following way: since $\mathrm{Ext}^\bullet_{\mathbb{P}^n \times \mathbb{P}^n}(\mathcal{O}(1,1), \mathcal{O}(k,0)) = H^\bullet(\mathbb{P}^n \times \mathbb{P}^n, h_1^*\mathcal{O}(k-1) \otimes h_2^*\mathcal{O}(-1))$, by the Leray spectral sequence we just need to prove that $h_{1*}(h_1^*\mathcal{O}(k-1) \otimes h_2^*\mathcal{O}(-1))$ is acyclic. By the projection formula one has

$$h_{1*}(h_1^*\mathcal{O}(k-1) \otimes h_2^*\mathcal{O}(-1)) = \mathcal{O}(k-1) \otimes h_{1*}h_2^*\mathcal{O}(-1) \tag{9.4.5}$$

and our claim follows by $h_{1*}h_2^*\mathcal{O}(-1) = 0$.

We are ready to explain the mutations. The main part of the process starts from the collection 9.4.2 and it can be described by n steps where the k^{th} step consists in:

- sending the first block of k bundles to the end by means of the inverse Serre functor, they get twisted by $\mathcal{O}(n+1, n+1)$

- the same block is orthogonal to the $n-k$ bundles at its left by Equation 9.4.4, hence we can move it $n-k$ steps to the left

After n steps we obtain the following collection:

$$\begin{aligned} D^b(M) = \langle \mathcal{O}(n,0) \ldots\ldots\ldots &\mathcal{O}(n,n) \\ \vdots \qquad\qquad &\vdots \\ \mathcal{O}(2n,n) \ldots\ldots\ldots &\mathcal{O}(2n,2n)\rangle \end{aligned} \tag{9.4.6}$$

and we can recover 9.4.3 with a twist by $\mathcal{O}(-n,-n)$. Note that by Equation 9.4.4 all mutations satisfy Condition (†), thus the proof is concluded. □

9.5 Derived equivalence for the roof of type A_4^G

Let V_5 be a vector space of dimension five. We recall the construction of the roof of type A_4^G:

$$\begin{tikzcd} & F(2,3,V_5) \arrow[dl, "h_1"'] \arrow[dr, "h_2"] & \\ G(2,V_5) & & G(3,V_5) \end{tikzcd} \tag{9.5.1}$$

In the following, as for the previous cases, we will call M the zero locus of a general section of $\mathcal{O}(1,1) = h_1^*\mathcal{O}(1) \otimes h_2^*\mathcal{O}(1)$, and the embedding of M in $F(2,3,V_5)$ will be denoted by l. We will call $\mathcal{U}_k$ and $\mathcal{Q}_k$ respectively the tautological and the quotient bundle of $G(k,V_5)$. We will use the minimal Lefschetz decomposition for $G(2,V_5)$ introduced in (Kuz08):

$$D^bG(2,V_5) = \left\langle \mathcal{O}, \mathcal{U}_2^\vee, \mathcal{O}(1), \mathcal{U}_2^\vee(1), \mathcal{O}(2), \mathcal{U}_2^\vee(2), \mathcal{O}(3), \mathcal{U}_2^\vee(3), \mathcal{O}(4), \mathcal{U}_2^\vee(4) \right\rangle \tag{9.5.2}$$

The duality isomorphism between $G(2,V_5)$ and $G(3,V_5)$ exchanges $\mathcal{U}_2^\vee$ with Q_3 and allows us to write a minimal Lefschetz exceptional collection for $G(3,V_5)$:

$$D^bG(3,V_5) = \langle \mathcal{O}, Q_3, \mathcal{O}(1), Q_3(1), \mathcal{O}(2), Q_3(2), \mathcal{O}(3), Q_3(3), \mathcal{O}(4), Q_3(4) \rangle\,. \tag{9.5.3}$$

Now, before explaining the mutations which will lead to our derived equivalence, let us prove some cohomology calculations which will be needed to perform such mutations. For the sake of brevity, in the remainder of this section, we will omit pullbacks to M while denoting exceptional objects of $D^b(M)$, but we will always keep track of the variety where Exts are computed.

Lemma 9.5.1. *The following relation holds for every integer k and for non negative integers a, b which satisfy $1 + a \leq b \leq 4 + a$ except for $b = 2 + a$:*

$$\mathrm{Ext}^\bullet_M(Q_3(k, k+b), \mathcal{O}(1+k, 1+k+a)) = 0$$

Proof. The proof is an application of Borel–Weil–Bott theorem. In particular, in light of the Koszul resolution of M, we are interested in understanding on which conditions on a and b we can obtain:

$$\begin{aligned} H^0(F, Q_3^\vee(1, 2+a-b)) &= 0 \\ H^0(F, Q_3^\vee(0, 1+a-b)) &= 0 \end{aligned} \tag{9.5.4}$$

Due to the Leray spectral sequence, our problem simplifies to showing that the pushforward of these bundles with respect to one of the two projections from the flag has no cohomology.

Namely, due to the projection formula, we have:

$$\begin{aligned} h_{2*}\mathcal{Q}_3^\vee(1, 2+a-b) = \mathcal{U}_3(1) \otimes \mathcal{Q}_3^\vee(2+a-b) = \wedge^2\mathcal{U}_3^\vee \otimes \mathcal{Q}_3^\vee(2+a-b) \\ = \wedge^2\mathcal{U}_3^\vee \otimes \left(\wedge^3\mathcal{U}_3^\vee\right)^{\otimes(2+a)} \otimes \mathcal{Q}_3^\vee \otimes \left(\wedge^2\mathcal{Q}_3^\vee\right)^{\otimes b} \end{aligned}$$

The Borel–Weil–Bott theorem states that the cohomology of $\wedge^2\mathcal{U}_3^\vee \otimes \left(\wedge^3\mathcal{U}_3^\vee\right)^{\otimes(2+a)}$ vanishes in every degree if two or more of the following integers coincide:

$$8+a;\ \ 7+a;\ \ 5+a;\ \ 3+b;\ \ 1+b.$$

while the cohomology of $\mathcal{Q}_3^\vee(0, 1+a-b)$ vanishes for $-5 \leq a-b \leq -1$ and this completes the proof. □

A similar result can be obtained with the same argument:

Lemma 9.5.2. *The following relation holds for every $k \in \mathbb{Z}$ and for non negative integers a, b which satisfy $3+a \leq b \leq 6+a$:*

$$\mathrm{Ext}^\bullet_M(\mathcal{O}(1+k, b+k), \mathcal{O}(2+k, 2+a+k)) = 0$$

We also prove the following statement, which will be necessary to verify Condition (†):

Lemma 9.5.3. *For every integer k and for non negative integers a, b which satisfy $1+a \leq b \leq 4+a$ one has:*

$$\begin{aligned} \mathrm{Ext}^\bullet_F(\mathcal{Q}_3(k+1, k+1+b), \mathcal{O}(k+1, k+1+a)) = 0 \\ \mathrm{Ext}^\bullet_F(\mathcal{O}(k+1, k+1+a), \mathcal{Q}_3(k, k+b)) = 0. \end{aligned} \tag{9.5.5}$$

Moreover, for every integer k and non negative integers a, b which satisfy $3 + a \leq b \leq 6 + a$ one has:

$$\begin{aligned} \operatorname{Ext}^\bullet_F(\mathcal{O}(2+k, b+1+k), \mathcal{O}(2+k, 2+a+k)) = 0 \\ \operatorname{Ext}^\bullet_F(\mathcal{O}(2+k, 2+a+k), \mathcal{O}(1+k, b+k)) = 0. \end{aligned} \tag{9.5.6}$$

Proof. All the claims are proven by semiorthogonality of the following full exceptional collection, which follows from an application of Theorem 3.3.1:

$$\begin{aligned} D^b(F) = \langle &\mathcal{O}(0,0), \mathcal{Q}_3(0,0), \dots\dots\dots, \mathcal{O}(0,4), \mathcal{Q}_3(0,4), \\ &\mathcal{O}(1,1), \mathcal{Q}_3(1,1), \dots\dots\dots, \mathcal{O}(1,5), \mathcal{Q}_3(1,5), \\ &\mathcal{O}(2,2), \mathcal{Q}_3(2,2), \dots\dots\dots, \mathcal{O}(2,6), \mathcal{Q}_3(2,6)\rangle. \end{aligned} \tag{9.5.7}$$

□

Another useful vanishing condition comes from the Leray spectral sequence and the Koszul resolution of M:

Lemma 9.5.4. *Let E_1 and E_2 be vector bundles on F such that they are pullbacks of vector bundles on $G(2, V_5)$. Then the following relation holds for every a, b, c, d such that $d - b = -1$:*

$$\operatorname{Ext}^\bullet_F(E_1(a,b), E_2(c,d)) = \operatorname{Ext}^\bullet_M(E_1(a,b), E_2(c,d)) = 0 \tag{9.5.8}$$

Moreover, if $\operatorname{Ext}^\bullet_F(E_2(c+1, d+1), E_1(a,b)) = \operatorname{Ext}^\bullet_M(E_2(c,d), E_1(a,b)) = 0$ *one has:*

$$\operatorname{Ext}^\bullet_F(E_2(c+d), E_1(a+b)) = 0. \tag{9.5.9}$$

The same result holds if E_1 and E_2 are pullbacks from $G(3, V_5)$ for every a, b, c, d such that $c - a = -1$.

Proof. By the Koszul resolution of M, we need to compute the cohomology of $E_1^\vee(-a,-b) \otimes E_2(c,d)$ and $E_1^\vee(-a,-b) \otimes E_2(c,d) \otimes \mathcal{O}(-1,-1)$ on F. We observe that

$$\begin{aligned} h_{1*}(E_1^\vee(-a,-b) \otimes E_2(c,d)) &= h_{1*}(h_1^*(E_1^\vee \otimes E_2(c-a)) \otimes h_2^*\mathcal{O}(d-b)) \\ &= E_1^\vee \otimes E_2(c-a) \otimes h_{1*}h_2^*\mathcal{O}(d-b) \end{aligned} \tag{9.5.10}$$

where the second equality follows by the projection formula, and similarly

$$\begin{aligned} &h_{1*}(E_1^\vee(-a,-b) \otimes E_2(c,d) \otimes \mathcal{O}(-1,-1)) = \\ &= E_1^\vee \otimes E_2(c-a-1) \otimes h_{1*}h_2^*\mathcal{O}(d-b-1). \end{aligned} \tag{9.5.11}$$

These bundles are zero for $d-b=-1$ because both $h_{1*}h_2^*\mathcal{O}(-1)$ and $h_{1*}h_2^*\mathcal{O}(-2)$ are zero, hence the cohomology of $E_1^\vee(-a,-b) \otimes E_2(c,d)$ vanishes by the Leray spectral sequence.

About Equation 9.5.9, the claim follows from the long exact sequence of cohomology associated to the Koszul resolution of $E_2^\vee \otimes E_1(a-c,b-d)$:

$$\begin{aligned} H^0(F, E_2^\vee \otimes E_1(a-c-1,b-d-1)) &\hookrightarrow H^0(F, E_2^\vee \otimes E_1(a-c,b-d)) \longrightarrow \\ H^0(M, E_2^\vee \otimes E_1(a-c,b-d)) &\longrightarrow H^1(F, E_2^\vee \otimes E_1(a-c-1,b-d-1)) \longrightarrow \cdots \end{aligned} \tag{9.5.12}$$

The proof for the statement about pullbacks from $G(3,V_5)$ is identical. $\square$

Lemma 9.5.5. *We have the following mutations in the derived category of M for every choice of the integers a,b and for $k \in \{1;2\}$:*

$$\begin{aligned} \mathbb{L}_{\mathcal{O}(a,b)}\mathcal{U}_k^\vee(a,b) &= \mathcal{Q}_k^\vee(a,b) \\ \mathbb{R}_{\mathcal{O}(a,b)}\mathcal{Q}_k^\vee(a,b) &= \mathcal{U}_k^\vee(a,b) \end{aligned} \tag{9.5.13}$$

and they satisfy Condition (†).

Proof. Let us compute $\mathbb{L}_{\mathcal{O}(a,b)}\mathcal{U}_k^\vee(a,b)$. The following result

$$\operatorname{Ext}^\bullet_M(\mathcal{O}(a,b),\mathcal{U}_k^\vee(a,b)) = V_5^\vee[0] \tag{9.5.14}$$

follows from the Borel–Weil–Bott theorem, it tells us that the mutation we are interested in is the cone of the morphism

$$V_5^\vee \otimes \mathcal{O}(a,b) \longrightarrow \mathcal{U}_k^\vee(a,b). \tag{9.5.15}$$

From the dual of the universal sequence

$$0 \longrightarrow \mathcal{U} \longrightarrow V_5 \otimes \mathcal{O} \longrightarrow \mathcal{Q} \longrightarrow 0 \tag{9.5.16}$$

we see that the morphism is surjective, thus the cone yields the kernel $\mathcal{Q}_k^\vee(a,b)$. The mutation $\mathbb{R}_{\mathcal{O}(a,b)}\mathcal{Q}_k^\vee(a,b)$ follows from an identical argument.

Both mutations satisfies Condition (†) because of the relations:

$$\begin{aligned} \operatorname{Ext}^\bullet_F(\mathcal{O},\mathcal{U}_k^\vee) &= \operatorname{Ext}^\bullet_F(\mathcal{U}_k^\vee(1,1),\mathcal{O}) = 0 \\ \operatorname{Ext}^\bullet_F(\mathcal{Q}_k^\vee,\mathcal{O}) &= \operatorname{Ext}^\bullet_F(\mathcal{O}(1,1),\mathcal{Q}_k^\vee) \ = 0 \end{aligned} \tag{9.5.17}$$

which can be verified by the Borel–Weil–Bott theorem. □

Lemma 9.5.6. *In the derived category of M, for every a and b one has the following mutations, which satisfy Condition (†):*

$$\begin{aligned} \mathbb{R}_{\mathcal{O}(a+1,b-1)}\mathcal{Q}_3(a,b) &= \mathcal{Q}_2(a,b) \\ \mathbb{R}_{\mathcal{O}(a+1,b-1)}\mathcal{U}_3(a,b) &= \mathcal{U}_2(a,b) \\ \mathbb{L}_{\mathcal{O}(a-1,b+1)}\mathcal{Q}_3^\vee(a,b) &= \mathcal{Q}_2^\vee(a,b) \\ \mathbb{L}_{\mathcal{O}(a-1,b+1)}\mathcal{U}_3^\vee(a,b) &= \mathcal{U}_2^\vee(a,b). \end{aligned} \tag{9.5.18}$$

Proof. With the Borel–Weil–Bott theorem we have:

$$\begin{aligned} \mathrm{Ext}^\bullet_M(\mathcal{Q}_3(a,b), \mathcal{O}(a+1,b-1)) &= \mathbb{C}[-1] \\ \mathrm{Ext}^\bullet_F(\mathcal{Q}_3(a+1,b+1), \mathcal{O}(a+1,b-1)) &= 0 \\ \mathrm{Ext}^\bullet_F(\mathcal{O}(a+1,b-1), \mathcal{Q}_3(a,b)) &= 0 \end{aligned} \tag{9.5.19}$$

so $\mathbb{R}_{\mathcal{O}(a+1,b-1)}\mathcal{Q}_3(a,b)$ is an extension and satisfies Condition (†). The relevant exact sequence is

$$0 \longrightarrow \mathcal{O}(1,-1) \longrightarrow \mathcal{Q}_2 \longrightarrow \mathcal{Q}_3 \longrightarrow 0, \tag{9.5.20}$$

which can be found computing the rank one cokernel of the injection $\mathcal{U}_2 \hookrightarrow \mathcal{U}_3$, comparing the universal sequences of the two Grassmannians and applying the Snake Lemma, this proves our first claim.
In order to verify the second one, we write the sequence involving the injection between the universal bundles, which is

$$0 \longrightarrow \mathcal{U}_2 \longrightarrow \mathcal{U}_3 \longrightarrow \mathcal{O}(1,-1) \longrightarrow 0. \tag{9.5.21}$$

The related Ext, in this case, is $\mathbb{C}[0]$, so the mutation is the cone of the morphism $\mathcal{U}_3 \longrightarrow \mathcal{O}(1,-1)$, yielding the desired result. Condition (†) is satisfied due to the following cohomological results:

$$\begin{aligned} \mathrm{Ext}^\bullet_F(\mathcal{U}_3(a+1,b+1), \mathcal{O}(a+1,b-1)) &= 0 \\ \mathrm{Ext}^\bullet_F(\mathcal{O}(a+1,b-1), \mathcal{U}_3(a,b)) &= 0. \end{aligned} \tag{9.5.22}$$

The proof for the last two mutations follow from the same arguments applied to the duals of Equations 9.5.21 and 9.5.20 and similar cohomological computations. □

Now we are ready to introduce the following result, which is the key of the proof of the derived equivalence.

Proposition 9.5.7. *Let Y_1 and Y_2 be the zero loci of the pushforwards of a general $s \in H^0(F, \mathcal{O}(1,1))$. Then there is a composition of mutations satisfying Condition (†), which yields an equivalence of categories*

$$\begin{aligned}\left\langle \phi_2 D^b(Y_2), D^b(G(3,V_5)) \otimes \mathcal{O}(1,1), D^b(G(3,V_5)) \otimes \mathcal{O}(2,2)\right\rangle \xrightarrow{\sim} \\ \left\langle \psi D^b(Y_2), D^b(G(2,V_5)) \otimes \mathcal{O}(1,1), D^b(G(2,V_5)) \otimes \mathcal{O}(2,2)\right\rangle\end{aligned} \tag{9.5.23}$$

where ψ is given by a composition of mutations and $\phi_2 = k_{2*}v_2^*$ in the notation of Diagram 9.1.1. Moreover, Y_1 and Y_2 are derived equivalent.

Proof. The idea of the proof is writing a full exceptional collection for M in a way such that we can use our cohomology vanishing results to transport line bundles $\mathcal{O}(a+1, b-1)$ to the immediate right of $\mathcal{Q}_3(a,b)$, then use Lemma 9.5.6 to get rid of $\mathcal{Q}_3(a,b)$, thus transforming pullbacks of vector bundles on $G(2,V_5)$ to pullbacks of vector bundles on $G(3,V_5)$.

We start from:

$$\begin{aligned}&D^b(M) = \langle \psi_1 D^b(Y_2),\\ &\mathcal{O}, \mathcal{Q}_3, \mathcal{O}(0,1), \mathcal{Q}_3(0,1), \mathcal{O}(0,2), \mathcal{Q}_3(0,2), \mathcal{O}(0,3), \mathcal{Q}_3(0,3),\\ &\mathcal{O}(0,4), \mathcal{Q}_3(0,4), \mathcal{O}(1,1), \mathcal{Q}_3(1,1), \mathcal{O}(1,2), \mathcal{Q}_3(1,2), \mathcal{O}(1,3),\\ &\mathcal{Q}_3(1,3), \mathcal{O}(1,4), \mathcal{Q}_3(1,4), \mathcal{O}(1,5), \mathcal{Q}_3(1,5)\rangle\end{aligned}$$

which is obtained by applying the Cayley trick to the collection 9.5.3 for $D^b(G(3,V_5))$, then twisting the whole decomposition by $\mathcal{O}(-1,-1)$. We defined ψ_1 as $\psi_1 = \phi_2(-) \otimes \mathcal{O}(-1,-1)$.

Our first operation is moving the first five bundles past $\psi_1 D^b(Y_2)$, then sending them to the end: they get twisted by the anticanonical bundle of M, which, with the adjunction formula, can be shown to be $\omega_M^\vee = \mathcal{O}(2,2)$.

$$\begin{aligned}
&D^b(M) = \langle \psi_2 D^b(Y_2),\\
&Q_3(0,2), \mathcal{O}(0,3), Q_3(0,3), \mathcal{O}(0,4), Q_3(0,4), \mathcal{O}(1,1), Q_3(1,1),\\
&\mathcal{O}(1,2), Q_3(1,2), \mathcal{O}(1,3), Q_3(1,3), \mathcal{O}(1,4), Q_3(1,4), \mathcal{O}(1,5),\\
&Q_3(1,5), \mathcal{O}(2,2), Q_3(2,2), \mathcal{O}(2,3), Q_3(2,3), \mathcal{O}(2,4)\rangle
\end{aligned}$$

where we introduced the functor

$$\psi_2 = \mathbb{R}_{\langle \mathcal{O}(0,0), Q_3(0,0), \mathcal{O}(0,1), Q_3(0,1), \mathcal{O}(0,2)\rangle}\psi_1 \tag{9.5.24}$$

Applying Lemma 9.5.1, we observe that $\mathcal{O}(1,1)$ can be moved next to $Q_3(0,2)$ and these mutations satisfy Condition (†) by Lemma 9.5.3. Then we can use Lemma 9.5.6 transforming $Q_3(0,2)$ in $Q_2(0,2)$. This can be done twice due to the invariance of the operation up to overall twists, yielding:

$$\begin{aligned}
&D^b(M) = \langle \psi_2 D^b(Y_2),\\
&\mathcal{O}(1,1), Q_2(0,2), \mathcal{O}(0,3), Q_3(0,3), \mathcal{O}(0,4), Q_3(0,4), Q_3(1,1),\\
&\mathcal{O}(1,2), Q_3(1,2), \mathcal{O}(1,3), \mathcal{O}(2,2), Q_2(1,3), \mathcal{O}(1,4), \mathcal{O}(2,3),\\
&Q_3(1,4), \mathcal{O}(1,5), Q_3(1,5), Q_3(2,2), Q_3(2,3), \mathcal{O}(2,4)\rangle
\end{aligned}$$

The next step is to move $\mathcal{O}(1,2)$ one step to the left. Since $Q_3(1,1) \simeq Q_3^\vee(1,2)$ the result follows from Lemma 9.5.5. Then, since by Lemma 9.5.1 and 9.5.2 $\mathcal{O}(1,2)$ is orthogonal to $\mathcal{O}(0,4)$ and $Q_3(0,4)$ we apply

Lemma 9.5.6 to transform $\mathcal{Q}_3(0,3)$ in $\mathcal{Q}_2(0,3)$. Again, all these operations can be performed twice (by invariance to overall twist) and they fulfill Condition (†):

$$\begin{aligned}
&D^b(M) = \langle \psi_2 D^b(Y_2),\\
&\mathcal{O}(1,1), \mathcal{Q}_2(0,2), \mathcal{O}(0,3), \mathcal{O}(1,2), \mathcal{Q}_2(0,3), \mathcal{O}(0,4), \mathcal{Q}_3(0,4),\\
&\mathcal{U}_3^\vee(1,2), \mathcal{Q}_3(1,2), \mathcal{O}(1,3), \mathcal{O}(2,2), \mathcal{Q}_2(1,3), \mathcal{O}(1,4), \mathcal{O}(2,3),\\
&\mathcal{Q}_2(1,4), \mathcal{O}(1,5), \mathcal{Q}_3(1,5), \mathcal{U}_3^\vee(2,3), \mathcal{Q}_3(2,3), \mathcal{O}(2,4)\rangle.
\end{aligned}$$

By Lemma 9.5.2 let us move $\mathcal{O}(0,3)$ one step to the right. Then, by a similar application of the Borel–Weil–Bott theorem we observe that $\mathcal{U}_3^\vee(1,2)$ is orthogonal to the two bundles at its left, hence we can move it to the immediate right of $\mathcal{Q}_2(0,4)$. These operations fulfill Condition (†) due to the following vanishings:

$$\begin{aligned}
\mathrm{Ext}^\bullet_F(\mathcal{U}_3(1,2), \mathcal{Q}_3(0,4)) &= 0\\
\mathrm{Ext}^\bullet_F(\mathcal{U}_3(1,2), \mathcal{O}(0,4)) &= 0\\
\mathrm{Ext}^\bullet_F(\mathcal{Q}_3(1,5), \mathcal{U}_3(1,2)) &= 0\\
\mathrm{Ext}^\bullet_F(\mathcal{O}(1,5), \mathcal{U}_3(1,2)) &= 0.
\end{aligned} \tag{9.5.25}$$

We can then move $\mathcal{U}_3^\vee(1,2)$ one additional step to the left using Lemma 9.5.6. Applying the same sequence of mutations to the $\mathcal{O}(1,1)$-twist of these objects we get the following collection:

$$\begin{aligned}
&D^b(M) = \langle \psi_2 D^b(Y_2),\\
&\mathcal{O}(1,1), \mathcal{Q}_2(0,2), \mathcal{O}(1,2), \mathcal{U}_2(0,3), \mathcal{U}_2^\vee(1,2), \mathcal{O}(0,3), \mathcal{O}(0,4),\\
&\mathcal{Q}_3(0,4), \mathcal{Q}_3(1,2), \mathcal{O}(1,3), \mathcal{O}(2,2), \mathcal{Q}_2(1,3), \mathcal{O}(2,3), \mathcal{U}_2(1,4),\\
&\mathcal{U}_2^\vee(2,3), \mathcal{O}(1,4), \mathcal{O}(1,5), \mathcal{Q}_3(1,5), \mathcal{Q}_3(2,3), \mathcal{O}(2,4)\rangle.
\end{aligned}$$

Again, thanks to Lemma 9.5.5, one has $\mathbb{R}_{\mathcal{O}(1,3)}\mathcal{Q}_3(1,2) \simeq \mathcal{U}_3^\vee(1,3)$, then we can apply Lemma 9.5.6 to transform $\mathcal{Q}_3(0,4)$ in $\mathcal{Q}_2(0,4)$. But then $\mathcal{O}(1,3)$ ends up next to $\mathcal{O}(0,4)$, which is orthogonal to it by application of Lemma 9.5.2, so they can be exchanged. Passing through $\mathcal{Q}_2(0,4)$ via Lemma 9.5.5 and mutating it to $\mathcal{U}_2(0,4)$, $\mathcal{O}(0,4)$ goes right next to $\mathcal{U}_3^\vee(1,3)$, which is mutated to $\mathcal{U}_2^\vee(1,3)$ by applying Lemma 9.5.6. Again all these mutations satisfy Condition (†).

Once we have done the same for the $\mathcal{O}(1,1)$-twists, we have transformed all the rank 2 and rank 3 pullbacks from $G(3,V_5)$ in pullbacks from $G(2,V_5)$. Removing all the duals by the identity $\mathcal{Q}_2 \simeq \wedge^2\mathcal{Q}_2^\vee \otimes \det(\mathcal{Q}_2)$ and the analogous one for $\mathcal{U}_2$, we get the following result:

$$
\begin{aligned}
&D^b(M) = \langle \psi_2 D^b(Y_2),\\
&\mathcal{O}(1,1), \mathcal{Q}_2(0,2), \mathcal{O}(1,2), \mathcal{U}_2(0,3), \mathcal{U}_2(2,2), \mathcal{O}(0,3), \mathcal{O}(1,3),\\
&\mathcal{U}_2(0,4), \mathcal{U}_2(2,3), \mathcal{O}(0,4), \mathcal{O}(2,2), \mathcal{Q}_2(1,3), \mathcal{O}(2,3), \mathcal{U}_2(1,4),\\
&\mathcal{U}_2(3,3), \mathcal{O}(1,4), \mathcal{O}(2,4), \mathcal{U}_2(1,5), \mathcal{U}_2(3,4), \mathcal{O}(1,5)\rangle.
\end{aligned}
\tag{9.5.26}
$$

First we send $\mathcal{O}(1,1)$ to the end, then we use Lemma 9.5.4 and the fact that $\mathcal{O}(2,-2)$ and $\mathcal{U}_2^\vee(2,-2)$ are acyclic to order the bundles by their power of the second twist:

$$
\begin{aligned}
&D^b(M) = \langle \psi_3 D^b(Y_2),\\
&\mathcal{Q}_2(0,2), \mathcal{O}(1,2), \mathcal{U}_2(2,2), \mathcal{O}(2,2), \mathcal{U}_2(0,3), \mathcal{O}(0,3), \mathcal{O}(1,3),\\
&\mathcal{U}_2(2,3), \mathcal{Q}_2(1,3), \mathcal{O}(2,3), \mathcal{U}_2(3,3), \mathcal{O}(3,3)\mathcal{U}_2(0,4), \mathcal{O}(0,4),\\
&\mathcal{U}_2(1,4), \mathcal{O}(1,4), \mathcal{O}(2,4), \mathcal{U}_2(3,4), \mathcal{U}_2(1,5), \mathcal{O}(1,5)\rangle
\end{aligned}
\tag{9.5.27}
$$

where we defined

$$\psi_3 = \mathbb{R}_{\mathcal{O}(1,1)}\psi_2. \tag{9.5.28}$$

Note that each time we exchanged objects $E_1(a,b), E_2(c,d)$ because $d-b=-1$ in order to get the collection 9.5.27, such mutations satisfy Condition (†) by Lemma 9.5.4 because $E_2(c+1,d+1)$ is semiorthogonal to $E_1(a,b)$. This semiorthogonality can be easily checked in the collection 9.5.26 (using the Serre functor if necessary).

Now we send the last 10 objects to the beginning and reorder again the collection with respect to the second twist, obtaining the following:

$$\begin{aligned}
&D^b(M) = \langle \psi_4 D^b(Y_2),\\
&\mathcal{U}_2(1,1), \mathcal{O}(1,1), \mathcal{U}_2(-2,2), \mathcal{O}(-2,2), \mathcal{U}_2(-1,2), \mathcal{O}(-1,2), \mathcal{O}(0,2),\\
&\mathcal{U}_2(1,2), \mathcal{Q}_2(0,2), \mathcal{O}(1,2), \mathcal{U}_2(2,2), \mathcal{O}(2,2), \mathcal{U}_2(-1,3), \mathcal{O}(-1,3),\\
&\mathcal{U}_2(0,3), \mathcal{O}(0,3), \mathcal{O}(1,3), \mathcal{U}_2(2,3), \mathcal{Q}_2(1,3), \mathcal{O}(2,3)\rangle,
\end{aligned}$$

where

$$\psi_4 = \mathbb{L}_{\langle \mathcal{U}_2(1,1),\mathcal{O}(1,1),\mathcal{U}_2(-2,2),\mathcal{O}(-2,2),\mathcal{U}_2(-1,2),\mathcal{O}(-1,2),\mathcal{O}(0,2),\mathcal{U}_2(1,2),\mathcal{U}_2(-1,3),\mathcal{O}(-1,3)\rangle}\psi_3 \tag{9.5.29}$$

Now we observe that $\mathcal{Q}_2(0,2)$ is orthogonal to $\mathcal{U}_2(1,2)$, so they can be exchanged: this allows to mutate $\mathcal{Q}_2(0,2)$ to $\mathcal{U}_2(0,2)$ sending it one step to the left (this mutation satisfies Condition (†) by a simple Borel–Weil–Bott computation). After doing the same thing with $\mathcal{O}(1,1)$–twists of these bundles, the last steps are sending the first two bundles to the end and twisting everything by $\mathcal{O}(-1,-1)$. We get:

$$\begin{aligned}
&D^b(M) = \langle \psi_5 D^b(Y_2),\\
&\mathcal{U}_2(-3,1), \mathcal{O}(-3,1), \mathcal{U}_2(-2,1), \mathcal{O}(-2,1), \mathcal{U}_2(-1,1), \mathcal{O}(-1,1),\\
&\mathcal{U}_2(0,1), \mathcal{O}(0,1), \mathcal{U}_2(1,1), \mathcal{O}(1,1), \mathcal{U}_2(-2,2), \mathcal{O}(-2,2), \mathcal{U}_2(-1,2),\\
&\mathcal{O}(-1,2), \mathcal{U}_2(0,2), \mathcal{O}(0,2), \mathcal{U}_2(1,2), \mathcal{O}(1,2), \mathcal{U}_2(2,2), \mathcal{O}(2,2)\rangle
\end{aligned} \tag{9.5.30}$$

where we defined the functor

$$\psi_5 = \mathcal{T}(-1,-1)\mathbb{R}_{\langle \mathcal{U}_2(1,1), \mathcal{O}(1,1)\rangle}\psi_4 \tag{9.5.31}$$

where $\mathcal{T}(-1,-1)$ is the twist with $\mathcal{O}(-1,-1)$.

Observe that, by the isomorphism $\mathcal{U}_2 \simeq \mathcal{U}_2^\vee(-1)$ and the fact that $\omega_{G(2,V_5)} \simeq \mathcal{O}(-5)$, one has:

$$\begin{aligned}
D^b(G(2,V_5)) = &\langle \mathcal{U}_2(-4), \mathcal{O}(-4), \mathcal{U}_2(-3), \mathcal{O}(-3), \mathcal{U}_2(-2,0),\\
&\mathcal{O}(-2), \mathcal{U}_2(-1), \mathcal{O}(-1), \mathcal{U}_2(0), \mathcal{O}(0)\rangle
\end{aligned} \tag{9.5.32}$$

Hence the collection 9.5.30 has the form

$$D^b(M) = \langle \psi_5 D^b(Y_2), D^b(G(2,V_5))\otimes\mathcal{O}(1,1), D^b(G(2,V_5))\otimes\mathcal{O}(2,2)\rangle. \tag{9.5.33}$$

The proof is completed once we compare this last decomposition with the following one, obtained by Cayley trick:

$$D^b(M) = \langle \phi_1 D^b(Y_1), D^b(G(2,V_5))\otimes\mathcal{O}(1,1), D^b(G(2,V_5))\otimes\mathcal{O}(2,2)\rangle. \tag{9.5.34}$$

□

9.6 Derived equivalence for the roof of type G_2

The roof of type G_2 is the complete flag F of type G_2, which admits projections h_1 and h_2 to the two G_2-Grassmannians G/P_1 and G/P_2 of dimension five, where the former is a smooth quadric. As usual, we denote by M the smooth zero locus of a section of $\mathcal{O}(1,1) = h_1^*\mathcal{O}(1) \otimes h_2^*\mathcal{O}(1)$, embedded in F by a map l, and we call (Y_1, Y_2) the associated Calabi–Yau pair.

The main result of this section is due to Kuznetsov in the paper (Kuz18), where by a sequence of mutations on a suitable semiorthogonal decomposition of M, a derived equivalence for the associated Calabi–Yau pair has been given. What we summarize here is basically the same argument, with some minor variation to make the result compatible with Condition (†) and the content of Chapters 10 and 11. First, one has the following semiorthogonal decompositions for the quadric (Kap88):

$$D^b(G/P_1) = \langle \mathcal{O}, \mathcal{S}^\vee, \mathcal{O}(1), \mathcal{O}(2), \mathcal{O}(3), \mathcal{O}(4), \rangle \tag{9.6.1}$$

where $\mathcal{S}$ is the spinor bundle (see (Ott88) for a detailed description of such object). By Serre functor, once we observe that $\omega_{G/P_1} = \mathcal{O}(-5)$, we write:

$$D^b(G/P_1) = \langle \mathcal{O}(-3), \mathcal{O}(-2), \mathcal{O}(-1), \mathcal{O}, \mathcal{S}^\vee, \mathcal{O}(1) \rangle. \tag{9.6.2}$$

One has $\mathrm{Ext}^\bullet_{G/P_1}(\mathcal{O}, \mathcal{S}^\vee) \simeq \mathbb{C}^8[0]$ by the Borel–Weil–Bott theorem, and in light of the following exact sequence (Ott88, Theorem 2.8):

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{O}^{\oplus 8} \longrightarrow \mathcal{S}^\vee \longrightarrow 0 \tag{9.6.3}$$

we compute $\mathbb{L}_{\mathcal{O}}\mathcal{S}^\vee \simeq \mathcal{S}$, hence the following decomposition (Kuz18, Equation 6):

$$D^b(G/P_1) = \langle \mathcal{O}(-3), \mathcal{O}(-2), \mathcal{O}(-1), \mathcal{S}, \mathcal{O}, \mathcal{O}(1)\rangle. \tag{9.6.4}$$

On the other hand, for the second G_2-Grassmannian there is the full exceptional collection (Kuz06, Section 6.4):

$$D^b(G/P_2) = \langle \mathcal{O}, \mathcal{U}^\vee, \mathcal{O}(1), \mathcal{U}^\vee(1), \mathcal{O}(2), \mathcal{U}^\vee(2)\rangle \tag{9.6.5}$$

where $\mathcal{U}$ is the tautological bundle. By Serre functor (since $\omega_{G/P_2} \simeq \mathcal{O}(-3)$) and by the isomorphism $\mathcal{U}^\vee \simeq \mathcal{U}(1)$ we write:

$$D^b(G/P_2) = \langle \mathcal{O}(-1), \mathcal{U}, \mathcal{O}, \mathcal{U}^\vee, \mathcal{O}(1), \mathcal{U}^\vee(1)\rangle \tag{9.6.6}$$

By Cayley trick one has the following semiorthogonal decompositions:

$$\begin{aligned} D^b(M) = \langle \phi_1 D^b(Y_1), &\mathcal{O}_M(-2,1), \mathcal{O}_M(-1,1), \mathcal{O}_M(0,1), \\ &l^*\mathcal{S}(1,1), \mathcal{O}_M(1,1), \mathcal{O}_M(2,1)\rangle. \end{aligned} \tag{9.6.7}$$

$$\begin{aligned} D^b(M) = \langle \phi_2 D^b(Y_2), &\mathcal{O}_M(1,0), l^*\mathcal{U}(1,1), \mathcal{O}_M(1,1), \\ &l^*\mathcal{U}^\vee(1,1), \mathcal{O}_M(1,2), l^*\mathcal{U}^\vee(1,2)\rangle \end{aligned} \tag{9.6.8}$$

For later convenience, let us also write the following exceptional collections for F, which are an application of Theorem 3.3.1 to the two projective bundle structures of F:

$$\begin{aligned} D^b(F) &= \langle \mathcal{O}(-3,0), \mathcal{O}(-2,0), \mathcal{O}(-1,0), \mathcal{S}(0,0), \mathcal{O}(0,0), \mathcal{O}(1,0), \\ &\qquad \mathcal{O}(-2,1), \mathcal{O}(-1,1), \mathcal{O}(0,1), \mathcal{S}(1,1), \mathcal{O}(1,1), \mathcal{O}(2,1)\rangle \\ &= \langle \mathcal{O}(0,0), \mathcal{U}^\vee(0,0), \mathcal{O}(0,1), \mathcal{U}^\vee(0,1), \mathcal{O}(0,2), \mathcal{U}^\vee(0,2), \\ &\qquad \mathcal{O}(1,1), \mathcal{U}^\vee(1,1), \mathcal{O}(1,2), \mathcal{U}^\vee(1,2), \mathcal{O}(1,3), \mathcal{U}^\vee(1,3)\rangle. \end{aligned} \tag{9.6.9}$$

We are now ready to formulate the following result, as a corollary to (Kuz18, Theorem 5)

Corollary 9.6.1. *In the setting above, there is a sequence of mutations of exceptional objects of $D^b(M)$ satisfying Condition (†) and realizing a derived equivalence*

$$\begin{aligned} &\langle \phi_2 D^b(Y_2), \mathcal{O}_M(1,0), l^*\mathcal{U}(1,1), \mathcal{O}_M(1,1), \\ &\qquad l^*\mathcal{U}^\vee(1,1), \mathcal{O}_M(1,2), l^*\mathcal{U}^\vee(1,2)\rangle \\ &\longrightarrow \langle \psi D^b(Y_2), \mathcal{O}_M(-2,1), \mathcal{O}_M(-1,1), \mathcal{O}_M(0,1), \\ &\qquad l^*\mathcal{S}(1,1), \mathcal{O}_M(1,1), \mathcal{O}_M(2,1)\rangle. \end{aligned} \tag{9.6.10}$$

hence giving $D^b(Y_1) \simeq D^b(Y_2)$.

Proof. Since $\omega_M = \mathcal{O}_M(-1,-1)$ (Kuz18), by applying the Serre functor to the last six objects of the collection 9.6.8 we find:

$$\begin{aligned} D^b(M) = \langle &\mathcal{O}_M(0,-1), l^*\mathcal{U}(0,0), \mathcal{O}_M(0,0), \\ &l^*\mathcal{U}^\vee(0,0), \mathcal{O}_M(0,1), l^*\mathcal{U}^\vee(0,1), \phi_2 D^b(Y_2)\rangle \end{aligned} \tag{9.6.11}$$

which can also be derived directly by applying Orlov's blowup formula (Theorem 3.3.4) to the collection 9.6.6, since M is the blowup of G/P_1 in Y_1, as it is explained in (Kuz18).
The first step is to send $\mathcal{O}_M(0,1)$ and $l^*\mathcal{U}^\vee(0,1)$ to the beginning by Serre functor, we get:

$$\begin{aligned} D^b(M) = \langle &\mathcal{O}_M(-1,0), l^*\mathcal{U}^\vee(-1,0), \mathcal{O}_M(0,-1), \\ &l^*\mathcal{U}(0,0), \mathcal{O}_M(0,0), l^*\mathcal{U}^\vee(0,0), \psi_1 D^b(Y_2)\rangle \end{aligned} \tag{9.6.12}$$

where we defined $\psi_1 := \mathbb{L}_{\langle \mathcal{O}_M(0,1), l^*\mathcal{U}^\vee(0,1)\rangle}\phi_2$. The next step is to move $\mathcal{O}_M(0,-1)$ to the beginning of the collection. By the Koszul

resolution of M and the results of (Kuz18, Lemma 1, Corollary 2) one finds $\mathrm{Ext}^\bullet_F(\mathcal{O}(-1,0),\mathcal{O}(0,-1)) = \mathrm{Ext}^\bullet_M(\mathcal{O}_M(-1,0),\mathcal{O}_M(0,-1)) = 0$ and $\mathrm{Ext}^\bullet_F(\mathcal{U}^\vee(-1,0),\mathcal{O}(0,-1)) = \mathrm{Ext}^\bullet_M(l^*\mathcal{U}^\vee(-1,0),\mathcal{O}_M(0,-1)) = 0$. Therefore we have $\mathbb{L}_{\langle \mathcal{O}_M(-1,0),l^*\mathcal{U}^\vee(-1,0)\rangle}\mathcal{O}_M(0,-1) \simeq \mathcal{O}_M(0,-1)$, and by semiorthogonality of 9.6.9 these mutations satisfy Condition (†).
We get the following collection:

$$\begin{aligned} D^b(M) = \langle &\mathcal{O}_M(0,-1),\mathcal{O}_M(-1,0),l^*\mathcal{U}^\vee(-1,0),\\ &l^*\mathcal{U}(0,0),\mathcal{O}_M(0,0),l^*\mathcal{U}^\vee(0,0),\psi_1 D^b(Y_2)\rangle \end{aligned} \tag{9.6.13}$$

Again by (Kuz18, Lemma 1, Corollary 2) we have $\mathrm{Ext}^\bullet_F(\mathcal{U}^\vee(0,1),\mathcal{U}(0,0)) = 0$ and $\mathrm{Ext}^\bullet_M(l^*\mathcal{U}^\vee(-1,0),l^*\mathcal{U}(0,0)) = \mathbb{C}[-1]$, while by semiorthogonality of the collections 9.6.9 also the vanishing $\mathrm{Ext}^\bullet_F\, \mathcal{U}(0,0),(\mathcal{U}^\vee(-1,0)) = 0$ holds, hence the mutation $\mathbb{L}_{l^*\mathcal{U}^\vee(-1,0)}l^*\mathcal{U}(0,0)$ satisfies Condition (†).
The result is a rank four extension which is isomorphic to $\mathcal{S}$ (Kuz18, Lemma 4). We get:

$$\begin{aligned} D^b(M) = \langle &\mathcal{O}_M(0,-1),\mathcal{O}_M(-1,0),\mathcal{S}(0,0),\\ &l^*\mathcal{U}^\vee(-1,0),\mathcal{O}_M(0,0),l^*\mathcal{U}^\vee(0,0),\psi_1 D^b(Y_2)\rangle \end{aligned} \tag{9.6.14}$$

The next operation is to move the first bundle to the end by means of the inverse Serre functor, and then move $\psi_1 D^b(Y_2)$ one step to the right. We find:

$$\begin{aligned} D^b(M) = \langle &\mathcal{O}_M(-1,0),\mathcal{S}(0,0),l^*\mathcal{U}^\vee(-1,0),\\ &\mathcal{O}_M(0,0),l^*\mathcal{U}^\vee(0,0),\mathcal{O}_M(1,0),\psi_2 D^b(Y_2)\rangle \end{aligned} \tag{9.6.15}$$

where we defined $\psi_2 = \mathbb{R}_{\mathcal{O}_M(1,0)}\psi_1$.
By the results of (Kuz18, Lemma 1, Corollary 2) it follows that $\mathrm{Ext}^\bullet_F(\mathcal{U}^\vee(0,1),\mathcal{O}) = 0$ and $\mathrm{Ext}^\bullet_M(l^*\mathcal{U}^\vee(-1,0),\mathcal{O}_M) = \mathbb{C}[0]$, while one

finds $\mathrm{Ext}^\bullet_F(\mathcal{O}, \mathcal{U}^\vee(-1,0),) = 0$ as a consequence of the semiorthogonality of 9.6.9. Hence, we can define a mutation $\mathbb{R}_{l^*\mathcal{U}^\vee(-1,0)}\mathcal{O}_M$ which satisfies Condition (†). We can compute such mutations by means of the following short exact sequence (Kuz18, Equation 5):

$$0 \longrightarrow \mathcal{O}(-1,1) \longrightarrow \mathcal{U}^\vee \longrightarrow \mathcal{O}(1,0) \longrightarrow 0. \tag{9.6.16}$$

The result is $\mathbb{R}_{l^*\mathcal{U}^\vee(-1,0)}\mathcal{O}_M \simeq \mathcal{O}_M(-2,1)$. In the same way we find $\mathbb{R}_{l^*\mathcal{U}^\vee(0,0)}\mathcal{O}_M(1,0) \simeq \mathcal{O}_M(-1,1)$ and the semiorthogonal decomposition becomes:

$$\begin{aligned} D^b(M) = \langle &\mathcal{O}_M(-1,0), \mathcal{S}(0,0), \mathcal{O}_M(0,0), \\ &\mathcal{O}_M(-2,1), \mathcal{O}_M(1,0), \mathcal{O}_M(-1,1), \psi_2 D^b(Y_2)\rangle \end{aligned} \tag{9.6.17}$$

Observe that by the vanishing of the cohomology of $\mathcal{O}_M(3,-1)$, $\mathcal{O}(2,-2)$ and $\mathcal{O}(-3,0)$ we see that the mutation obtained by exchanging $\mathcal{O}_M(-2,1)$ with $\mathcal{O}_M(1,0)$ satisfies Condition (†). After applying such operation, let us move $\psi_2 D^b(Y_2)$ two steps to the left, then send the block $\mathcal{O}_M(-1,0), \mathcal{S}(0,0), \mathcal{O}_M(0,0), \mathcal{O}_M(1,0)$ to the end. Summing all up, we obtain:

$$\begin{aligned} D^b(M) = \langle \psi_3 D^b(Y_2), &\mathcal{O}_M(-2,1), \mathcal{O}_M(-1,1), \mathcal{O}_M(0,1), \\ &\mathcal{S}(1,1), \mathcal{O}_M(1,1), \mathcal{O}_M(2,1)\rangle \end{aligned} \tag{9.6.18}$$

where we introduced the functor $\psi_3 = \mathbb{R}_{\langle \mathcal{O}_M(-2,1), \mathcal{O}_M(-1,1)\rangle}\psi_2$. The proof is concluded by setting $\psi_3 = \psi$. □

10 Derived equivalence of Calabi–Yau fibrations

10.1 Setup and notation

The scope of this section is to provide a method to extend the equivalences of Chapter 9 to zero loci of pushforwards of general sections of $\mathcal{L}$ on a homogeneous roof bundle, where $\mathcal{L}$ is a basepoint-free line bundle such that for every $b \in B$ the restriction map $H^0(\mathcal{Z}, \mathcal{L}) \longrightarrow H^0(\pi^{-1}(b), \mathcal{L}|_{\pi^{-1}(b)})$ is surjective, and $L = \mathcal{L}|_{\pi^{-1}(b)}$ restricts to $\mathcal{O}(1)$ on both projective bundle structures of $\pi^{-1}(b) \simeq G/P$. More precisely, let us consider a homogeneous roof bundle $\mathcal{Z}$ of type G/P over a smooth projective base B, with the locally trivial fibration $\pi : \mathcal{Z} \longrightarrow B$. Fix a general section $\Sigma \in H^0(\mathcal{Z}, \mathcal{L})$ with smooth zero locus $\mathcal{M}$ and the corresponding pair of varieties $X_1 = Z(p_{1*}\Sigma)$, $X_2 = Z(p_{2*}\Sigma)$ (which are Calabi–Yau fibrations if $G/P \not\simeq \mathbb{P}^n \times \mathbb{P}^n$ by Lemma 4.4.4).

We have the following diagram:

$$
\begin{tikzcd}
 & \mathcal{T}_1 \arrow[r, hook, "m_1"] \arrow[ddl, "\mu_1"'] & \mathcal{M} \arrow[d, hook, "\iota"] & \mathcal{T}_2 \arrow[l, hook', "m_2"'] \arrow[ddr, "\mu_2"] & \\
 & & \mathcal{Z} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] \arrow[dd, "\pi"] & & \\
X_1 \arrow[r, hook, "u_1"] & \mathcal{Z}_1 \arrow[dr, "r_1"] & & \mathcal{Z}_2 \arrow[dl, "r_2"'] & X_2 \arrow[l, hook', "u_2"'] \\
 & & B & &
\end{tikzcd}
\tag{10.1.1}
$$

where for $i \in \{1;2\}$ $\mathcal{T}_i$ is a $\mathbb{P}^{r-1}$-bundle over X_i defined as the preimage of X_i under the map $\mathcal{M} \longrightarrow \mathcal{Z}_i$ given by restricting p_i to $\mathcal{M}$.

Remark 10.1.1. Consider $b \in Z(\pi_*\Sigma)$. Then one has $x \in \mathcal{M}$ for every $x \in \pi^{-1}(b)$, therefore $\bar{\pi}^{-1}(b) \simeq G/P$. Conversely, if $b \notin Z(\pi_*\Sigma)$, $\bar{\pi}^{-1}(b)$ is isomorphic to a (possibly singular) zero locus of a section of $L \simeq \mathcal{L}|_{\pi^{-1}(b)}$. This allows us to conclude that, for general Σ, the fiber of $\bar{\pi}$ over the general $b \in B$ has expected codimension.

Let us now choose a general $b \in B$ and call $M := \mathcal{M} \times_B \{b\}$. By the discussion of Remark 10.1.1, one has $M = Z(\sigma)$ where $\sigma = \Sigma|_{\pi^{-1}(b)}$ is a section of L. Let us choose b such that M is smooth. We recover the following diagram (Diagram 9.1.1):

$$
\begin{tikzcd}
 & T_1 \arrow[rr, hook, "k_1"] \arrow[ddl, "\nu_1"'] & & M \arrow[d, hook, "l"'] & & T_2 \arrow[ll, hook', "k_2"'] \arrow[ddr, "\nu_2"] & \\
 & & & G/P \arrow[dl, "h_1"'] \arrow[dr, "h_2"] & & & \\
Y_1 \arrow[r, hook, "t_1"] & G/P_1 & & & & G/P_2 & Y_2 \arrow[l, hook', "t_2"']
\end{tikzcd}
\tag{10.1.2}
$$

Moreover, for $i \in \{1;2\}$ the general fiber of $X_i \longrightarrow B$ is isomorphic to Y_i (Lemma 4.4.4). As we discussed in Section 9.1 one has the following semiorthogonal decompositions of $D^b(M)$:

$$D^b(M) \simeq \langle \theta_1 D^b(Y_1), l^* h_1^* D^b(G/P_1) \otimes L, \dots, l^* h_1^* D^b(G/P_1) \otimes L^{\otimes(r-1)} \rangle \tag{10.1.3}$$

$$\simeq \langle \theta_2 D^b(Y_2), l^* h_2^* D^b(G/P_2) \otimes L, \dots, l^* h_2^* D^b(G/P_2) \otimes L^{\otimes(r-1)} \rangle \tag{10.1.4}$$

where $\theta_i := k_{i*}\nu_i^*$ and r is the rank of the vector bundles whose projectivizations yield G/P.

10.1.1 Semiorthogonal decompositions for $\mathcal{M}$

Let us first observe that, since $\mathcal{M}$ is the zero locus of a smooth section of $\mathcal{L}$ and $\mathcal{Z}$ is a $\mathbb{P}^{r-1}$-bundle over both $\mathcal{Z}_1$ and $\mathcal{Z}_2$, by Cayley trick we have the following semiorthogonal decompositions:

$$\begin{aligned} D^b(\mathcal{M}) \simeq& \langle \phi_1 D^b(X_1), \iota^* p_1^* D^b(\mathcal{Z}_1) \otimes \mathcal{L}, \dots, \iota^* p_1^* D^b(\mathcal{Z}_1) \otimes \mathcal{L}^{\otimes(r-1)} \rangle \\ \simeq& \langle \phi_2 D^b(X_2), \iota^* p_2^* D^b(\mathcal{Z}_2) \otimes \mathcal{L}, \dots, \iota^* p_2^* D^b(\mathcal{Z}_2) \otimes \mathcal{L}^{\otimes(r-1)} \rangle \end{aligned} \tag{10.1.5}$$

where $\phi_i := m_{i*} \circ \mu_i^*$. Note that r is the same as in Equation 10.1.3.

The next step is to construct semiorthogonal decompositions for $\mathcal{Z}_i$. Let us assume there exist full exceptional collections

$$\begin{aligned} D^b(G/P_1) &= \langle J_1, \dots, J_m \rangle \\ D^b(G/P_2) &= \langle K_1, \dots, K_m \rangle. \end{aligned} \tag{10.1.6}$$

Note that on a homogeneous variety G/P every exceptional object is a homogeneous vector bundle if G is simply connected and semisimple (Böh06, Proposition 2.1.4). Therefore each object of the collections listed above is a homogeneous vector bundle of the form $J_i = G \times^{P_1} V_{\Gamma^{J_i}}$ for some representation Γ^{J_i} of P_1 acting on a vector space $V_{\Gamma^{J_i}}$. Let us fix a principal G-bundle $\mathcal{V} \longrightarrow B$ such that $\mathcal{Z} = \mathcal{V} \times^G G \times^P G/P$. Then, for every i we define the vector bundle $\mathcal{J}_i := \mathcal{V} \times^G G \times^{P_1} V_{\Gamma^{J_i}}$ on $\mathcal{Z}_1 = \mathcal{V} \times^G G \times^{P_1} G/P_1$, and we define some bundles $\mathcal{K}_i$ in the same way on $\mathcal{Z}_2$. In this way, we defined two sets of vector bundles $\{\mathcal{J}_1, \ldots, \mathcal{J}_n\} \subset D^b(\mathcal{Z}_1)$ and $\{\mathcal{K}_1, \ldots, \mathcal{K}_n\} \subset D^b(\mathcal{Z}_2)$ such that $J_i = \mathcal{J}_i|_{\pi^{-1}(b)}$ and $K_i = \mathcal{K}_i|_{\pi^{-1}(b)}$ for every $b \in B$.

If we now apply (Sam06, Theorem 3.1) to such bundles we obtain semiorthogonal decompositions for $D^b(Z_1)$ and $D^b(Z_2)$:

$$\begin{aligned} D^b(\mathcal{Z}_1) &= \langle \mathcal{J}_1 \otimes r_1^* D^b(B), \ldots, \mathcal{J}_m \otimes r_1^* D^b(B) \rangle \\ D^b(\mathcal{Z}_2) &= \langle \mathcal{K}_1 \otimes r_2^* D^b(B), \ldots, \mathcal{K}_m \otimes r_2^* D^b(B) \rangle. \end{aligned} \tag{10.1.7}$$

Then, substituting these decompositions in Equation 10.1.5 we have:

$$\begin{aligned} D^b(\mathcal{M}) \simeq \langle & \phi_1 D^b(X_1), \\ & \mathfrak{B} \otimes \iota^* p_1^* \mathcal{J}_1 \otimes \mathcal{L}, \ldots\ldots\ldots\ldots, \mathfrak{B} \otimes \iota^* p_1^* \mathcal{J}_n \otimes \mathcal{L}, \\ & \qquad \vdots \qquad\qquad\qquad\qquad\qquad \vdots \\ & \mathfrak{B} \otimes \iota^* p_1^* \mathcal{J}_1 \otimes \mathcal{L}^{\otimes(r-1)}, \ldots\ldots, \mathfrak{B} \otimes \iota^* p_1^* \mathcal{J}_N \otimes \mathcal{L}^{\otimes(r-1)} \rangle \\ \simeq \langle & \phi_2 D^b(X_2), \\ & \mathfrak{B} \otimes \iota^* p_2^* \mathcal{K}_1 \otimes \mathcal{L}, \ldots\ldots\ldots\ldots, \mathfrak{B} \otimes \iota^* p_2^* \mathcal{K}_N \otimes \mathcal{L}, \\ & \qquad \vdots \qquad\qquad\qquad\qquad\qquad \vdots \\ & \mathfrak{B} \otimes \iota^* p_2^* \mathcal{K}_1 \otimes \mathcal{L}^{\otimes(r-1)}, \ldots\ldots, \mathfrak{B} \otimes \iota^* p_2^* \mathcal{K}_n \otimes \mathcal{L}^{\otimes(r-1)} \rangle \end{aligned} \tag{10.1.8}$$

where we introduced the shorthand notation $\mathfrak{B} := \iota^* \pi^* D^b(B)$. We can compare Equation 10.1.8 with the following semiorthogonal decompositions for M, which can be found by substituting 10.1.6 in 10.1.3:

$$\begin{aligned}
D^b(M) \simeq \;& \langle \theta_1 D^b(Y_1), \\
& l^* h_1^* K_1 \otimes L, \dots\dots\dots\dots, l^* h_1^* K_N \otimes L, \\
& \qquad \vdots \qquad\qquad\qquad\qquad \vdots \\
& l^* h_1^* K_1 \otimes L^{\otimes(r-2)}, \dots\dots, l^* h_1^* K_N \otimes L^{\otimes(r-1)} \rangle \\
\simeq \;& \langle \theta_2 D^b(Y_2), \\
& l^* h_2^* \widetilde{K}_1 \otimes L, \dots\dots\dots\dots, l^* h_2^* \widetilde{K}_N \otimes L, \\
& \qquad \vdots \qquad\qquad\qquad\qquad \vdots \\
& l^* h_2^* \widetilde{K}_1 \otimes L^{\otimes(r-2)}, \dots\dots, l^* h_2^* \widetilde{K}_N \otimes L^{\otimes(r-1)} \rangle
\end{aligned} \tag{10.1.9}$$

The goal of this chapter is to prove that if there is a sequence of mutations identifying the semiorthogonal complements of $D^b(Y_1)$ and $D^b(Y_2)$ in the collections 10.1.9, under some requirements we will introduce below, there exists a sequence of mutations for the collections 10.1.8 providing a derived equivalence $D^b(X_1) \simeq D^b(X_2)$. Although this resembles a problem of base change, additional care is required: in fact, given a smooth section $\Sigma \in H^0(\mathcal{Z}, \mathcal{L})$, it is not assured that every restriction to the fibers over B gives rise to smooth varieties, or even varieties of the expected codimension as we discussed in Remark 10.1.1. Hence, we will need to introduce some additional hypotheses.

Note that the semiorthogonal decompositions 10.1.8 do not depend on the choice of $\Sigma \in H^0(\mathcal{Z}, \mathcal{L})$ as long as its zero locus is smooth. Hence, finding a sequence of mutations as above would prove derived equivalence for the pair (X_1, X_2) of Calabi–Yau fibrations defined as $X_i = Z(p_{i*}\Sigma)$ for *any* $\Sigma \in H^0(\mathcal{Z}, \mathcal{L})$ with smooth zero locus.

Remark 10.1.2. Before stating the main results, let us remind that in order to apply (Sam06, Theorem 3.1) to 10.1.5 it is required to have a full exceptional collection for the fibers of the locally trivial fibrations r_1 and r_2. The problem of finding full exceptional collections for homogeneous varieties is still open, but there are many cases where a solution has been found. Let G/P be a roof with projective bundle structures $h_i : G/P \longrightarrow G/P_i$ for $i \in \{1; 2\}$. Let us review the cases where a full exceptional collection is known for both G/P_1 and G/P_2.

- Type $A_n \times A_n$, A_n^M and A_{2n}^G: here G/P_i is a $SL(V)$-Grassmannian for some vector space V. Full exceptional collections for these varieties have been constructed in (Kap88).
- Type $C_{3n/2-1}$: in this case G/P_i is a symplectic Grassmannian. The only case where a full exceptional collection is known for both G/P_1 and G/P_2 is the roof of type C_2. The collections have been established in (Bei78; Kap88).
- Type D_n: the only two cases where both G/P_i have known full exceptional collections are D_4 and D_5. In the former, by triality, G/P_i is a six dimensional quadric, for which a full exceptional collection has been found in (Kap88). In the latter, the varieties G/P_i are spinor tenfolds, a full exceptional collection for them is given in (Kuz06).
- Type G_2: there are known full exceptional collections for both G/P_1 and G/P_2 (Kap88; Kuz06).
- Type F_4: To the best of the author's knowledge, no full exceptional collection is known for F_4/P^2 and F_4/P^3.

10.2 Preparatory material

Before stating the main results of this sections, we need to formulate a technical lemma. To this purpose we shall fix the notation and review some basic material, we refer to (Bo94, Chapter 3.1) for a more thorough introduction.

Definition 10.2.1. *A functor $L : \mathfrak{A} \longrightarrow \mathfrak{B}$ is* left adjoint *to a functor $R : \mathfrak{B} \longrightarrow \mathfrak{A}$ if there exists a natural transformation $\eta : I_{\mathfrak{A}} \Rightarrow RL$, called* unit*, such that for every $A \in \mathfrak{A}$:*

- *$\eta_A : A \longrightarrow RL(A)$ is a morphism of $\mathfrak{A}$, called* unit morphism *of A*
- *for every object A' of $\mathfrak{A}$ and for every morphism $a : A \longrightarrow R(A')$, there exists a unique $\alpha : L(A') \longrightarrow A'$ such that the following diagram commutes:*

$$\begin{tikzcd} & R(A') \\ A \arrow[ur, "a"] \arrow[r, "\eta_A"'] & RL(A) \arrow[u, dashed, "R(\alpha)"'] \end{tikzcd} \tag{10.2.1}$$

A functor $R : \mathfrak{B} \longrightarrow \mathfrak{A}$ is right adjoint to a functor $L : \mathfrak{A} \longrightarrow \mathfrak{B}$ if there is a natural transformation $\epsilon : LR \Rightarrow I_{\mathfrak{B}}$, called counit*, such that for every $B \in \mathfrak{B}$:*

- *$\epsilon_B : LR(B) \longrightarrow B$ is a morphism of $\mathfrak{B}$, called* counit morphism *of B*
- *for every object B' of $\mathfrak{B}$ and for every morphism $b : L(B') \longrightarrow B$, there exists a unique $\beta : L(A') \longrightarrow A'$ such that the following*

diagram commutes:

$$\begin{tikzcd} L(B') \arrow[d, dashed, "L(\alpha)"'] \arrow[dr, "g"] & \\ LR(B) \arrow[r, "\eta_B"'] & B \end{tikzcd} \tag{10.2.2}$$

Theorem 10.2.2. *(Bo94, Theorem 3.1.5) Consider two functors $L : \mathfrak{A} \longrightarrow \mathfrak{B}$ and $R : \mathfrak{B} \longrightarrow \mathfrak{A}$. Then, the following statements are equivalent:*

1. *L is left adjoint to R*

2. *R is right adjoint to L*

3. *There exist natural transformations $\eta : I_{\mathfrak{A}} \Rightarrow RL$ and $\epsilon : LR \Rightarrow I_{\mathfrak{B}}$, called* unit *and* counit*, such that for every object A of $\mathfrak{A}$ and B of $\mathfrak{B}$ one has the following relations, called* triangle identities:

$$\begin{aligned} I_{LA} &: LA \xrightarrow{L(\eta_A)} LRLA \xrightarrow{\epsilon_{LA}} LA \\ I_{RB} &: RB \xrightarrow{\eta_{RB}} RLRB \xrightarrow{R(\epsilon_B)} RB \end{aligned} \tag{10.2.3}$$

4. *For every pair of objects A, B respectively of $\mathfrak{A}$ and $\mathfrak{B}$, there is a bijection $\theta_{A,B}$: $\mathrm{Hom}_{\mathfrak{B}}(L(A), B)) \longrightarrow \mathrm{Hom}_{\mathfrak{A}}(A, R(B))$ which is natural in both A and B, i.e. for every $a \in \mathrm{Hom}_{\mathfrak{A}}(A', A)$ and*

$b \in \mathrm{Hom}_{\mathfrak{B}}(B, B')$ the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathfrak{B}}(L(A), B) \arrow[r, "\theta_{A,B}"] \arrow[d, "b\circ(-)\circ L(a)"'] & \mathrm{Hom}_{\mathfrak{A}}(A, R(B)) \arrow[d, "R(b)\circ(-)\circ a"] \\
\mathrm{Hom}_{\mathfrak{B}}(L(A'), B') \arrow[r, "\theta_{A',B'}"'] & \mathrm{Hom}_{\mathfrak{A}}(A', R(B'))
\end{tikzcd} \tag{10.2.4}$$

Hereafter we will denote such adjoint pairs by the symbol $L\colon \mathfrak{A} \rightleftarrows \mathfrak{B} : R$. As a consequence of the above we can prove the following lemma.

Lemma 10.2.3. *Let $\mathfrak{B}, \mathfrak{M}, \mathfrak{X}$ be categories. Consider the adjoint pairs $L_1\colon \mathfrak{X} \rightleftarrows \mathfrak{M} : R_1$ and $L_2\colon \mathfrak{B} \rightleftarrows \mathfrak{X} : R_2$. Then, for every object $x \in \mathfrak{X}$ one has the following identity:*

$$L_1(\epsilon_{2,x}) = \epsilon_{12,\ L_1x} \circ L_1L_2R_2(\eta_{1,x}). \tag{10.2.5}$$

where by ϵ_{12,L_1x} we denote the counit morphism of the adjoint pair $L_1L_2\colon \mathfrak{B} \rightleftarrows \mathfrak{M} : R_2R_1$ on the object $L_1x \in \mathfrak{M}$.

Proof. Let us call θ_1 and θ_2 the bijections of Theorem 10.2.2, part 3, for the adjoint pairs $L_1\colon \mathfrak{X} \rightleftarrows \mathfrak{M} : R_1$ and $L_2\colon \mathfrak{B} \rightleftarrows \mathfrak{X} : R_2$. To simplify the notation, we drop the subscripts denoting objects for such morphisms. Let us fix $\tau_i = \theta_i^{-1}$ for $i \in \{1; 2\}$. By Theorem 10.2.2, part

3, we have the following diagram where each square is commutative:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathfrak{B}}(R_2R_1L_1x, R_2R_1L_1x) \arrow[r, "(-)\circ R_2(\eta_{1,x})"] \arrow[d, "\tau_2"'] & \mathrm{Hom}_{\mathfrak{B}}(R_2x, R_2R_1L_1x) \arrow[d, "\tau_2"] \\
\mathrm{Hom}_{\mathfrak{X}}(L_2R_2R_1L_1x, R_1L_1x) \arrow[r, "(-)\circ L_2R_2(\eta_{1,x})"] & \mathrm{Hom}_{\mathfrak{X}}(L_2R_2x, R_1L_1x) \\
\mathrm{Hom}_{\mathfrak{M}}(L_1L_2R_2R_1L_1x, L_1x) \arrow[r, "(-)\circ L_1L_2R_2(\eta_{1,x})"] \arrow[u, "\theta_1"] & \mathrm{Hom}_{\mathfrak{M}}(L_1L_2R_2x, L_1x) \arrow[u, "\theta_1"']
\end{tikzcd} \tag{10.2.6}$$

On the other hand, one has the following diagram, where the upper square commutes:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathfrak{B}}(R_2x, R_2x) \arrow[r, "R_2(\eta_{1,x})\circ(-)"] \arrow[d, "\tau_2"'] & \mathrm{Hom}_{\mathfrak{B}}(R_2x, R_2R_1L_1x) \arrow[d, "\tau_2"] \\
\mathrm{Hom}_{\mathfrak{X}}(L_2R_2x, x) \arrow[r, "\eta_{1,x}\circ(-)"] \arrow[dr, "L_1(-)"'] & \mathrm{Hom}_{\mathfrak{X}}(L_2R_2x, R_1L_1x) \\
& \mathrm{Hom}_{\mathfrak{M}}(L_1L_2R_2x, L_1x) \arrow[u, "\theta_1"']
\end{tikzcd} \tag{10.2.7}$$

Let us now prove that the lower triangular diagram commutes as well, namely we want to show that given any morphism $f \in \mathrm{Hom}_{\mathfrak{X}}(L_2R_2x, x)$ one has $\theta_1(L_1(f)) = \eta_{1,x} \circ f$. Again by naturality of θ_1, as in Theorem

10.2.2, we have a commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathfrak{M}}(L_1x, L_1x) \arrow[r, "\theta_1"] \arrow[d, "(-)\circ L_1(f)"'] & \mathrm{Hom}_{\mathfrak{X}}(x, R_1L_1x) \arrow[d, "(-)\circ f"] \\
\mathrm{Hom}_{\mathfrak{M}}(L_1L_2R_2x, L_1x) \arrow[r, "\theta_1"] & \mathrm{Hom}_{\mathfrak{X}}(L_2R_2x, R_1L_1x)
\end{tikzcd} \tag{10.2.8}$$

Let us now consider the identity morphism $I_{L_1x} \in \mathrm{Hom}(L_1x, L_1x)$. Then, by composing the top and right arrows, one has $\theta_1(I_{L_1x}) \circ f = \eta_{1,x} \circ f$. On the other hand, by composition of the bottom and left arrows, we get $\theta_1(I_{L_1x} \circ L_1(f)) = \theta_1(L_1(f))$ and by commutativity we obtain $\theta_1(L_1(f)) = \eta_{1,x} \circ f$.

Now, given the identity morphisms $I_{R_2R_1L_1x} \in \mathrm{Hom}(R_2R_1L_1x, R_2R_1L_1x)$ and $I_{R_2x} \in \mathrm{Hom}(R_2x, R_2x)$, one has $I_{R_2R_1L_1x} \circ R_2(\eta_{1,x}) = R_2(\eta_{1,x}) \circ I_{R_2x}$, therefore $\tau_1(\tau_2(I_{R_2R_1L_1x} \circ R_2(\eta_{1,x})) = \tau_1(\tau_2(R_2(\eta_{1,x}) \circ I_{R_2x}))$. By commutativity of Diagram 10.2.6 one finds

$$\tau_1(\tau_2(I_{R_2R_1L_1x} \circ R_2(\eta_{1,x})) = \tau_1(\tau_2(I_{R_2R_1L_1x})) \circ L_1L_2R_2(\eta_{1,x}) \tag{10.2.9}$$

while by commutativity of diagram 10.2.7 it follows that

$$\tau_1(\tau_2(R_2(\eta_{1,x}) \circ I_{R_2x})) = L_1(\tau_2(I_{R_2x})). \tag{10.2.10}$$

The proof is concluded by equating the right-hand sides of Equations 10.2.9 and 10.2.10 once we observe that $\tau_2(I_{R_2x}) = \epsilon_{2,x}$ and $\tau_1(\tau_2(I_{R_2R_1L_1x})) = \epsilon_{12,L_1x}$. □

10.3 Roof bundles and mutations

In the following we will mostly focus on left mutations, since given an admissible subcategory $\mathfrak{A} \subset \mathfrak{C}$ of a triangulated category $\mathfrak{C}$, the functors $\mathbb{L}_{\mathfrak{A}} :^{\perp} \mathfrak{A} \longrightarrow \mathfrak{A}^{\perp}$ and $\mathbb{R}_{\mathfrak{A}} : \mathfrak{A}^{\perp} \longrightarrow^{\perp} \mathfrak{A}$ are mutually inverses (see, for example, (Kuz10, Lemma 2.7) and the source therein).

Definition 10.3.1. *Let $\pi : \mathcal{Z} \longrightarrow B$ be a flat proper morphism of smooth projective varieties and let $\mathcal{M}$ be the smooth zero locus of a section of a line bundle $\mathcal{L}$, embedded in $\mathcal{Z}$ by $\iota : \mathcal{M} \hookrightarrow \mathcal{Z}$. Consider two relatively exceptional objects $\mathcal{E}, \mathcal{F} \in D^b(\mathcal{Z})$ and suppose there exist strong semiorthogonal decompositions*

$$\begin{aligned} D^b(\mathcal{Z}) &= \langle C, \mathcal{E} \otimes \pi^* D^b(B), \mathcal{F} \otimes \pi^* D^b(B) \rangle \\ D^b(\mathcal{M}) &= \langle \mathcal{D}, \iota^*\mathcal{E} \otimes \iota^*\pi^* D^b(B), \iota^*\mathcal{F} \otimes \iota^*\pi^* D^b(B) \rangle \end{aligned} \tag{10.3.1}$$

for some admissible subcategories $C, \mathcal{D}$. We say that the left mutation $\mathbb{L}_{\langle \mathcal{E} \otimes \pi^* D^b(B) \rangle}(\mathcal{F} \otimes \pi^* D^b(B))$ commutes *with ι^* if the following equivalence holds:*

$$\begin{aligned} D^b(\mathcal{M}) &= \langle \mathcal{D}, \mathbb{L}_{\langle \iota^*\mathcal{E} \otimes \iota^*\pi^* D^b(B) \rangle}(\iota^*\mathcal{F} \otimes \iota^*\pi^* D^b(B)), \iota^*\mathcal{E} \otimes \iota^*\pi^* D^b(B) \rangle \\ &\simeq \langle \mathcal{D}, \iota^*\mathbb{L}_{\langle \mathcal{E} \otimes \pi^* D^b(B) \rangle}(\mathcal{F} \otimes \pi^* D^b(B)), \iota^*\mathcal{E} \otimes \iota^*\pi^* D^b(B) \rangle \end{aligned} \tag{10.3.2}$$

Definition 10.3.2. *Let $\mathcal{Z} \longrightarrow B$ be a flat and proper morphism of smooth projective varieties, let $\mathcal{L}$ be a line bundle on $\mathcal{Z}$. Consider two objects $\mathcal{E}, \mathcal{F} \in D^b(\mathcal{Z})$. We say that $\mathcal{E}$ is $\mathcal{L}$-*semiorthogonal *to $\mathcal{F}$ if the following condition is fulfilled:*

$$\pi_* R\mathcal{H}om_{\mathcal{Z}}(\mathcal{E}, \mathcal{F} \otimes \mathcal{L}^\vee) = 0 \tag{10.3.3}$$

Lemma 10.3.3. *In the setting of Definition 10.3.2, let B be a point. Then $\mathcal{E}$ is $\mathcal{L}$-semiorthogonal to $\mathcal{F}$ if and only if* $\mathrm{Ext}^\bullet_{\mathcal{Z}}(\mathcal{E}, \mathcal{F} \otimes \mathcal{L}^\vee) = 0$.

Proof. Fix $B \simeq \{pt\}$. One has:

$$\begin{aligned} \pi_* R\mathcal{H}om_{G/P}(\mathcal{E}, \mathcal{F} \otimes \mathcal{L}^\vee) &\simeq H^\bullet(G/P, R\mathcal{H}om_{G/P}(\mathcal{E}, \mathcal{F} \otimes \mathcal{L}^\vee)) \\ &\simeq \mathrm{Ext}^\bullet_{G/P}(\mathcal{E}, \mathcal{F} \otimes \mathcal{L}^\vee) \end{aligned} \tag{10.3.4}$$

where the first isomorphism is a consequence of (Mum12, Page 50, Corollary 2). ◻

Lemma 10.3.4. *Let $\pi : \mathcal{Z} \longrightarrow B$ be a flat and proper morphism of smooth projective varieties, call $\mathcal{M} \subset \mathcal{Z}$ the smooth zero locus of a section of a line bundle $\mathcal{L}$, embedded in $\mathcal{Z}$ by the morphism ι. Suppose there exist admissible subcategories $C \subset D^b(\mathcal{M})$, $\mathcal{D} \subset D^b(\mathcal{Z})$ and vector bundles $\mathcal{E}, \mathcal{F} \in D^b(\mathcal{Z})$ relatively exceptional over B such that one has the following strong, B-linear semiorthogonal decompositions:*

$$\begin{aligned} D^b(\mathcal{Z}) &= \langle \mathcal{D}, \mathcal{E} \otimes \pi^* D^b(B), \mathcal{F} \otimes \pi^* D^b(B) \rangle \\ D^b(\mathcal{M}) &= \langle C, \iota^*\mathcal{E} \otimes \iota^*\pi^* D^b(B), \iota^*\mathcal{F} \otimes \iota^*\pi^* D^b(B) \rangle \end{aligned} \tag{10.3.5}$$

Then, if $\mathcal{E}$ is $\mathcal{L}$-semiorthogonal to $\mathcal{F}$, $\mathbb{L}_{\langle \mathcal{E} \otimes \pi^* D^b(B) \rangle} \mathcal{F} \otimes \pi^* D^b(B)$ commutes with ι^*.

Proof. In order to describe the left mutation of $\iota^*\mathcal{F} \otimes \pi^*\mathcal{G}$ through $\langle \iota^*\mathcal{E} \otimes \iota^*\pi^* D^b(B) \rangle$ inside $D^b(\mathcal{M})$, we introduce the following functors (and their right adjoints):

$$\begin{aligned} \Psi_{\langle \mathcal{E} \otimes \pi^* D^b(B) \rangle} : D^b(B) &\longrightarrow D^b(\mathcal{Z}) \\ \mathcal{G} &\longmapsto \pi^*\mathcal{G} \otimes \mathcal{E} \end{aligned} \tag{10.3.6}$$

$$\begin{aligned} \Psi^!_{\langle \mathcal{E}\otimes\pi^* D^b(B)\rangle} : D^b(\mathcal{Z}) &\longrightarrow D^b(B) \\ \mathcal{R} &\longmapsto \pi_* R\mathcal{H}om_{\mathcal{Z}}(\mathcal{E},\mathcal{R}) \end{aligned} \tag{10.3.7}$$

We can apply Lemma 10.2.3 to the data $L_1 = \iota^*$, $R_1 = \iota_*$, $L_2 = \Psi_{\langle \mathcal{E}\otimes\pi^* D^b(B)\rangle}$, $R_2 = \Psi^!_{\langle \mathcal{E}\otimes\pi^* D^b(B)\rangle}$, $\mathfrak{X} = D^b(\mathcal{Z})$, $\mathfrak{B} = D^b(B)$, $\mathfrak{M} = D^b(\mathcal{M})$. As a result, the following diagram commutes:

$$\begin{tikzcd} L_1L_2R_2A \arrow[r, "L_1(\epsilon_{2,A})"] \arrow[d, "L_1L_2R_2(\eta_{1,A})"'] & L_1A \arrow[d, equal] \\ L_1L_2R_2R_1L_1A \arrow[r, "\epsilon_{12,L_1A}"] & L_1A \end{tikzcd} \tag{10.3.8}$$

for any object A of $D^b(\mathcal{Z})$. Let us now introduce the following functors:

$$\begin{aligned} \Theta_{\langle \iota^*\mathcal{E}\otimes\iota^*\pi^* D^b(B)\rangle} : D^b(B) &\longrightarrow D^b(\mathcal{M}) \\ \mathcal{G} &\longmapsto \iota^*\pi^*\mathcal{G}\otimes\iota^*\mathcal{E} \end{aligned} \tag{10.3.9}$$

$$\begin{aligned} \Theta^!_{\langle \iota^*\mathcal{E}\otimes\iota^*\pi^* D^b(B)\rangle} : D^b(\mathcal{M}) &\longrightarrow D^b(B) \\ \mathcal{W} &\longmapsto \pi_*\iota_* R\mathcal{H}om_{\mathcal{M}}(\iota^*\mathcal{E},\mathcal{W}) \end{aligned} \tag{10.3.10}$$

Note that, by definition, $\Theta_{\langle \iota^*\mathcal{E}\otimes\iota^*\pi^* D^b(B)\rangle} = \iota^*\Psi_{\langle \mathcal{E}\otimes\pi^* D^b(B)\rangle}$. Then, Diagram 10.3.8 can be written in the following way:

$$\begin{tikzcd}
\iota^*\Psi_{\langle\mathcal{E}\otimes\pi^*D^b(B)\rangle}\Psi^!_{\langle\mathcal{E}\otimes\pi^*D^b(B)\rangle}(\mathcal{F}\otimes\pi^*\mathcal{G}) \arrow[r, "\iota^*\epsilon_{\mathcal{Z}}"] \arrow[d, "\alpha"] & \iota^*(\mathcal{F}\otimes\pi^*\mathcal{G}) \arrow[d, equal] \\
\iota^*\Psi_{\langle\mathcal{E}\otimes\pi^*D^b(B)\rangle}\Psi^!_{\langle\mathcal{E}\otimes\pi^*D^b(B)\rangle}\iota_*\iota^*(\mathcal{F}\otimes\pi^*\mathcal{G}) \arrow[r, "\epsilon_{\mathcal{M}}"'] & \iota^*(\mathcal{F}\otimes\pi^*\mathcal{G})
\end{tikzcd} \tag{10.3.11}$$

where $\alpha = \iota^*\Psi_{\langle\mathcal{E}\otimes\pi^*D^b(B)\rangle}\Psi^!_{\langle\mathcal{E}\otimes\pi^*D^b(B)\rangle}(\eta_{1,\mathcal{F}\otimes\pi^*\mathcal{G}})$, $\epsilon_{\mathcal{Z}} = \epsilon_{2,\mathcal{F}\otimes\pi^*\mathcal{G}}$, $\epsilon_{\mathcal{M}} = \epsilon_{12,\iota^*(\mathcal{F}\otimes\pi^*\mathcal{G})}$.

Let us now prove the following claim:

Claim. *The map α is an isomorphism if $\mathcal{E}$ is $\mathcal{L}$-semiorthogonal to $\mathcal{F}$.*

To this purpose, let us focus on the following term:

$$\Psi^!_{\langle\mathcal{E}\otimes\pi^*D^b(B)\rangle}\iota_*\iota^*(\mathcal{F}\otimes\pi^*\mathcal{G}) = \pi_*R\mathcal{H}om_{\mathcal{Z}}(\mathcal{E},\iota_*\iota^*(\mathcal{F}\otimes\pi^*\mathcal{G})) \tag{10.3.12}$$

Since $\mathcal{E}$ is a vector bundle we have:

$$\pi_*R\mathcal{H}om_{\mathcal{Z}}(\mathcal{E},\iota_*\iota^*(\mathcal{F}\otimes\pi^*\mathcal{G})) \simeq \pi_*(\mathcal{E}^\vee\otimes\iota_*\iota^*(\mathcal{F}\otimes\pi^*\mathcal{G})). \tag{10.3.13}$$

Observe that $\iota_*\iota^*(\mathcal{F}\otimes\pi^*\mathcal{G})$ has a resolution given by the tensor product of $\mathcal{F}$ with the Koszul resolution of $\iota_*\iota^*\mathcal{O}$:

$$0\longrightarrow \mathcal{F}\otimes\pi^*\mathcal{G}\otimes\mathcal{L}^\vee\longrightarrow\mathcal{F}\otimes\pi^*\mathcal{G}\longrightarrow\iota_*\iota^*(\mathcal{F}\otimes\pi^*\mathcal{G})\longrightarrow 0 \tag{10.3.14}$$

By left-exactness of the derived pushforward we get the following long exact sequence:

$$\begin{aligned}0 \longrightarrow R^0\pi_*(\mathcal{E}^\vee \otimes \mathcal{F} \otimes \pi^*\mathcal{G} \otimes \mathcal{L}^\vee) \longrightarrow R^0\pi_*(\mathcal{E}^\vee \otimes \mathcal{F} \otimes \pi^*\mathcal{G}) \longrightarrow \\ \longrightarrow R^0\pi_*(\mathcal{E}^\vee \otimes \iota_*\iota^*(\mathcal{F} \otimes \pi^*\mathcal{G})) \longrightarrow R^1\pi_*(\mathcal{E}^\vee \otimes \mathcal{F} \otimes \pi^*\mathcal{G} \otimes \mathcal{L}^\vee) \longrightarrow \\ \longrightarrow R^1\pi_*(\mathcal{E}^\vee \otimes \mathcal{F} \otimes \pi^*\mathcal{G}) \longrightarrow R^1\pi_*(\mathcal{E}^\vee \otimes \iota_*\iota^*(\mathcal{F} \otimes \pi^*\mathcal{G})) \longrightarrow \cdots\end{aligned} \tag{10.3.15}$$

Hence, proving the claim reduces to show that

$$R^k\pi_*(\mathcal{E}^\vee \otimes \mathcal{F} \otimes \pi^*\mathcal{G} \otimes \mathcal{L}^\vee) = 0 \tag{10.3.16}$$

for every k. By the (derived) projection formula one has:

$$\begin{aligned}R^k\pi_*(\mathcal{E}^\vee \otimes \mathcal{F} \otimes \pi^*\mathcal{G} \otimes \mathcal{L}^\vee) &\simeq R^k\pi_*(\mathcal{E}^\vee \otimes \mathcal{F} \otimes \mathcal{L}^\vee) \otimes \mathcal{G} \\ &\simeq R^k\pi_* R\mathcal{H}om_{\mathcal{Z}}(\mathcal{E}, \mathcal{F} \otimes \mathcal{L}^\vee) \otimes \mathcal{G} = 0\end{aligned} \tag{10.3.17}$$

where the last isomorphism follows by the fact that $\mathcal{E}$ is $\mathcal{L}$-semiorthogonal to $\mathcal{F}$. This proves that α is an isomorphism for every $\mathcal{G} \in D^b(B)$.

In order to prove that $\iota^*\mathbb{L}_{\langle \mathcal{E}\otimes\pi^*D^b(B)\rangle}(\mathcal{F}\otimes\pi^*\mathcal{G})$ and $\mathbb{L}_{\langle \iota^*\mathcal{E}\otimes\iota^*\pi^*D^b(B)\rangle}\iota^*(\mathcal{F}\otimes \pi^*\mathcal{G})$ are isomorphic for every $\mathcal{G} \in D^b(B)$, note that such objects are defined as the cones of respectively $\epsilon_{\mathcal{M}}$ and $\iota^*\epsilon_{\mathcal{Z}}$, in the following distinguished triangles:

$$\begin{aligned}\Theta_{\langle \iota^*\mathcal{E}\otimes\iota^*\pi^*D^b(B)\rangle}\Theta^!_{\langle \iota^*\mathcal{E}\otimes\iota^*\pi^*D^b(B)\rangle}\iota^*\mathcal{F} \otimes \iota^*\pi^*\mathcal{G} \xrightarrow{\epsilon_{\mathcal{M}}} \\ \longrightarrow \iota^*\mathcal{F} \otimes \iota^*\pi^*\mathcal{G} \longrightarrow \mathbb{L}_{\langle \iota^*\mathcal{E}\otimes\iota^*\pi^*D^b(B)\rangle}\iota^*\mathcal{F} \otimes \iota^*\pi^*\mathcal{G};\end{aligned} \tag{10.3.18}$$

$$\begin{aligned}\iota^*\Psi_{\langle \mathcal{E}\otimes\pi^*D^b(B)\rangle}\Psi^!_{\langle \mathcal{E}\otimes\pi^*D^b(B)\rangle}\mathcal{F} \otimes \pi^*\mathcal{G} \xrightarrow{\iota^*\epsilon_{\mathcal{Z}}} \\ \longrightarrow \iota^*(\mathcal{F} \otimes \pi^*\mathcal{G}) \longrightarrow \iota^*\mathbb{L}_{\langle \mathcal{E}\otimes\pi^*D^b(B)\rangle}\mathcal{F} \otimes \pi^*\mathcal{G}.\end{aligned}$$

Since α is an isomorphism, the proof is concluded by (GM03, page 232, Corollary 4) applied to Diagram 10.3.11.

□

With the following lemma, in the setting of a roof bundle $\pi : \mathcal{Z} \longrightarrow B$, we show that $\mathcal{L}$-semiorthogonality of vector bundles constructed by a representation of P can be checked on the fibers of π.

Lemma 10.3.5. *Let $\pi : \mathcal{Z} \longrightarrow B$ be a roof bundle of type G/P where B is a smooth projective variety and $\mathcal{V}$ a principal G-bundle on B. Let Γ be a representation of P acting on the vector space V_Γ such that $L = G \times^P V_\Gamma$. Consider a line bundle $\mathcal{L}$ on $\mathcal{Z}$ such that $\mathcal{L} = \pi^* T \otimes (\mathcal{V} \times^G G \times^P V_\Gamma)$ where T is a line bundle on B. Let $\mathcal{E}, \mathcal{F}$ be vector bundles on $\mathcal{Z}$ such that for every $b \in B$ one has $\mathcal{E}|_{\pi^{-1}(b)} \simeq E$ and $\mathcal{F}|_{\pi^{-1}(b)} \simeq F$, where E is L-semiorthogonal to F. Then, for every $\mathcal{G} \in D^b(B)$, $\mathcal{E}$ is $\mathcal{L}$-semiorthogonal to $\mathcal{F} \otimes \pi^* \mathcal{G}$.*

Proof. Since $\mathcal{E}$ is a vector bundle, the following holds:

$$\pi_* R\mathcal{H}om_{\mathcal{Z}}(\mathcal{E}, \mathcal{F} \otimes \pi^* \mathcal{G} \otimes \mathcal{L}^\vee) \simeq \pi_*(\mathcal{E}^\vee \otimes \mathcal{F} \otimes \pi^* \mathcal{G} \otimes \mathcal{L}^\vee) \quad (10.3.19)$$

thus our claim follows by proving that $R^k \pi_*(\mathcal{E}^\vee \otimes \mathcal{F} \otimes \pi^* \mathcal{G} \otimes \mathcal{L}^\vee) = 0$ for every k. By the derived projection formula one has:

$$R^k \pi_*(\mathcal{E}^\vee \otimes \mathcal{F} \otimes \pi^* \mathcal{G} \otimes \mathcal{L}^\vee) = R^k \pi_*(\mathcal{E}^\vee \otimes \mathcal{F} \otimes \mathcal{L}^\vee) \otimes \mathcal{G}. \quad (10.3.20)$$

We will prove that the stalk $R^k \pi_*(\mathcal{E}^\vee \otimes \mathcal{F} \otimes \mathcal{L}^\vee)_b$ vanishes for every $b \in B$. Once we fix b, we observe that by our assumptions the following holds:

- $\mathcal{E}^\vee \otimes \mathcal{F} \otimes \mathcal{L}^\vee$ is flat over B (Har77, Proposition III.9.2)
- The map $b \longmapsto \dim H^k(\pi^{-1}(b), \mathcal{E}^\vee \otimes \mathcal{F} \otimes \mathcal{L}^\vee|_{\pi^{-1}(b)}) = \dim H^k(G/P, E^\vee \otimes F \otimes L^\vee)$ is constant for every k

Then, we apply (Mum12, Page 50, Corollary 2) and we find:

$$\begin{aligned} R^k\pi_*(\mathcal{E}^\vee \otimes \mathcal{F} \otimes \mathcal{L}^\vee)_b &\simeq H^k(G/P, E^\vee \otimes F \otimes L^\vee) \\ &\simeq \operatorname{Ext}^k_{G/P}(E, F \otimes L^\vee) = 0 \end{aligned} \tag{10.3.21}$$

where the last equality holds because E is L-semiorthogonal to F by hypothesis. This proves that $R^k\pi_*(\mathcal{E}^\vee \otimes \mathcal{F} \otimes \mathcal{L}^\vee) = 0$, hence concluding the proof. □

Notation 10.3.6. *We establish the following data:*

- *Consider a homogeneous roof bundle $\mathcal{Z} \xrightarrow{\pi} B$ of type G/P on a smooth projective base B, such that $\mathcal{Z}$ admits two different projective bundle structures $p_i : \mathcal{Z} \longrightarrow \mathcal{Z}_i$ for $i \in \{1; 2\}$. Call $\mathcal{V}$ a principal G-bundle over B such that $\mathcal{Z} = \mathcal{V} \times^G G/P$.*
- *Let L be the line bundle on G/P which restricts to $\mathcal{O}(1)$ on the fibers of both the projective bundle structures of G/P (which exists by Proposition 4.1.4). Given a representation Γ of P acting on the vector space V_Γ such that $L = G \times^P V_\Gamma$, consider a line bundle $\mathcal{L}$ on $\mathcal{Z}$ such that $\mathcal{L} = \pi^*T \otimes (\mathcal{V} \times^G G \times^P V_\Gamma)$ where T is a line bundle on B, and such that $\mathcal{L}$ restricts to $\mathcal{O}(1)$ on each fiber of both the projective bundle structures of $\mathcal{Z}$.*
- *A general section $\Sigma \in H^0(\mathcal{Z}, \mathcal{L})$ with smooth zero locus $\mathcal{M}$, a general section $\sigma \in H^0(G/P, L)$, a point $b \in B$ such that $M :=$*

$Z(\sigma) = Z(\Sigma|_{\pi^{-1}(b)})$ is smooth, and the following diagram:

$$\begin{tikzcd} M = \mathcal{M} \times_B \{b\} \arrow[r] \arrow[d] & \mathcal{M} \arrow[d] \\ \{b\} \arrow[r] & B \end{tikzcd} \tag{10.3.22}$$

Call $\iota : \mathcal{M} \hookrightarrow \mathcal{Z}$ and $l : M \hookrightarrow G/P$ the respective embeddings.

- *A Calabi–Yau pair (Y_1, Y_2) associated to the roof G/P, defined as $Y_i = Z(h_{i*}\sigma_i)$ where $h_i : G/P \longrightarrow G/P_i$ are the two projective bundle structures of G/P. Similarly, we consider a pair (X_1, X_2) of Calabi–Yau fibrations defined as $X_i = Z(p_{i*}\Sigma)$.*

- *Two full exceptional collections:*

$$D^b(G/P_1) = \langle J_1, \dots, J_m \rangle, \quad D^b(G/P_2) = \langle K_1, \dots, K_m \rangle \tag{10.3.23}$$

which by (Orl92, Corollary 2.7) induce the following semiorthogonal decompositions for $D^b(G/P)$ (recall that G/P has $\mathbb{P}^{r-1}$-bundle structures on both G/P_1 and G/P_2):

$$D^b(G/P) = \langle E_1, \dots, E_N \rangle = \langle F_1, \dots, F_N \rangle \tag{10.3.24}$$

where the bundles E_i have the form $E_i = J_j \otimes L^{\otimes k}$ and $F_i = K_j \otimes L^{\otimes k}$ for some integers j, k. Moreover, by Theorem (Orl03 Proposition 2.10), one gets the following semiorthogonal decompositions for $D^b(M)$:

$$\begin{aligned} D^b(M) &= \langle \theta_1 D^b(Y_1), l^* E_1, \dots, l^* E_n \rangle \\ &= \langle \theta_2 D^b(Y_2), l^* F_1, \dots, l^* F_n \rangle \end{aligned} \tag{10.3.25}$$

where θ_1 and θ_2 are defined in Equation 10.1.9. Note that the numbers of exceptional objects in Equations 10.3.24 and 10.3.25 are different, but all exceptional objects of 10.3.25 are pullbacks of objects from 10.3.24. We introduce the following ordered lists of exceptional objects:

$$\begin{aligned} \boldsymbol{E} &= (E_1, \ldots, E_n) \\ \boldsymbol{F} &= (F_1, \ldots, F_n) \end{aligned} \tag{10.3.26}$$

Observe that $\boldsymbol{E}$ and $\boldsymbol{F}$ are subsets of the generators of $D^b(G/P)$ appearing in the exceptional collections 10.3.24.

- *For every J_i, which can be described as $J_i = G \times^{P_1} V_{\Gamma_i}$ for some vector space V_{Γ_i} associated to a representation Γ_i of P_1, consider the vector bundle $\mathcal{J}_i := \mathcal{V} \times^G G \times^{P_i} V_{\Gamma_i}$ such as defined in Section 10.1.1, with the property that $\mathcal{J}_i|_{\pi^{-1}(b)} \simeq J_i$ for every i. Then, by (Sam06, Theorem 3.1), the subcategory $\mathcal{J}_i \otimes r_1^* D^b(B)$ is admissible in $D^b(\mathcal{Z}_1)$. In the same way we can define admissible subcategories $\mathcal{K}_i \otimes r_2^* D^b(B) \subset D^b(\mathcal{Z}_2)$. By applying (Sam06, Theorem 3.1) and (Orl92, Corollary 2.7) to the collections 10.3.25 we find the following semiorthogonal decompositions:*

$$\begin{aligned} D^b(\mathcal{Z}) &= \langle [E_1], \ldots, [E_N] \rangle \\ &= \langle [F_1], \ldots, [F_N] \rangle \end{aligned} \tag{10.3.27}$$

where the subcategories $[E_i]$ have the form $[E_i] = p_1^* \mathcal{J}_i \otimes \mathcal{L}^{\otimes k} \otimes \pi^* D^b(B)$. From now on, given a homogeneous vector bundle $G \times^P V_\Gamma$ on G/P, we will use the notation $[G \times^P V_\Gamma]$ to denote $(\mathcal{V} \times^G G \times^P V_\Gamma) \otimes \pi^* D^b(B) \subset \mathcal{Z}$. Since L and J_i are both homogeneous, *and by definition of $\mathcal{L}$, one has $[E_i] \simeq [h_1^* J_i \otimes L^{\otimes k}]$. Similarly, $[F_i] = [h_2^* K_j \otimes L^{\otimes k}]$. Furthermore, by (Orl03 Proposition 2.10), there are semiorthogonal decompositions:*

$$\begin{aligned} D^b(\mathcal{M}) &= \langle \phi_1 D^b(X_1), \iota^*[E_1], \ldots, \iota^*[E_n] \rangle \\ &= \langle \phi_2 D^b(X_2), \iota^*[F_1], \ldots, \iota^*[F_n] \rangle \end{aligned} \tag{10.3.28}$$

where ϕ_1 and ϕ_2 are the same as in Equation 10.1.5.

◦ *Consider a sequence of pairs $(\boldsymbol{E}^{(\lambda)}, \psi^{(\lambda)})$ where, for each λ, $\boldsymbol{E}^{(\lambda)} = \left(E_1^{(\lambda)}, \ldots, E_n^{(\lambda)}\right)$ is an ordered list of exceptional objects of $D^b(G/P)$ and $\psi^{(\lambda)} : D^b(Y_1) \longrightarrow D^b(M)$ is a fully faithful functor, and a sequence of operations:*

$$\xi^{(\lambda)} : (\boldsymbol{E}^{(\lambda)}, \psi^{(\lambda)}) \longmapsto (\boldsymbol{E}^{(\lambda+1)}, \psi^{(\lambda+1)}) \tag{10.3.29}$$

such that each $\xi^{(\lambda)}$ falls in one of the following classes (type O1, O2 or O3):

O1 *For $1 \leq i \leq n-1$ exchanging the order of $E_{i+1}^{(\lambda)}$ with $E_i^{(\lambda)}$ and substituting the latter with $\mathbb{L}_{E_i} E_{i+1}^{(\lambda)}$, while leaving $\psi^{(\lambda)}$ unchanged:*

$$\begin{aligned} \left(E_1^{(\lambda+1)}, \ldots, E_n^{(\lambda+1)}\right) &= \Big(E_1^{(\lambda)}, \ldots, E_{i-1}^{(\lambda)}, \mathbb{L}_{E_i^{(\lambda)}} E_{i+1}^{(\lambda)}, \\ &\qquad E_i^{(\lambda)}, E_{i+2}^{(\lambda)}, \ldots, E_n^{(\lambda)}\Big) \\ \psi^{(\lambda+1)} &= \psi^{(\lambda)} \end{aligned} \tag{10.3.30}$$

where $\mathbb{L}_{E_i^{(\lambda)}} E_{i+1}^{(\lambda)}$ is defined by the following distinguished triangle in $D^b(G/P)$:

$$E_i^{(\lambda)} \otimes \operatorname{Ext}^\bullet_{G/P}(E_i^{(\lambda)}, E_{i+1}^{(\lambda)}) \longrightarrow E_{i+1}^{(\lambda)} \longrightarrow \mathbb{L}_{E_i^{(\lambda)}} E_{i+1}^{(\lambda)}. \tag{10.3.31}$$

Similarly, we define the operation of exchanging the order of $E_i^{(\lambda)}$ with $E_{i+1}^{(\lambda)}$ and substituting the former with $\mathbb{R}_{E_{i+1}^{(\lambda)}} E_i^{(\lambda)}$, while leaving $\psi^{(\lambda)}$ unchanged:

$$\begin{aligned}\left(E_1^{(\lambda+1)}, \ldots, E_n^{(\lambda+1)}\right) &= \Big(E_1^{(\lambda)}, \ldots, E_{i-1}^{(\lambda)}, E_{i+1}^{(\lambda)}, \mathbb{R}_{E_{i+1}^{(\lambda)}} E_i^{(\lambda)}, \\ &\qquad E_{i+2}^{(\lambda)}, \ldots, E_n^{(\lambda)}\Big) \\ \psi^{(\lambda+1)} &= \psi^{(\lambda)}\end{aligned} \tag{10.3.32}$$

where $\mathbb{R}_{E_{i+1}^{(\lambda)}} E_i^{(\lambda)}$ is defined by the following distinguished triangle in $D^b(G/P)$:

$$\mathbb{R}_{E_{i+1}^{(\lambda)}} E_i^{(\lambda)} \longrightarrow E_i^{(\lambda)} \longrightarrow E_{i+1}^{(\lambda)} \otimes \mathrm{Ext}^\bullet_{G/P}(E_i^{(\lambda)}, E_{i+1}^{(\lambda)})^\vee. \tag{10.3.33}$$

O2 *Sending $E_1^{(\lambda)}$ to the end, twisting it by $L^{\otimes(r-1)}$ and substituting $\psi^{(\lambda)}$ with $\mathbb{R}_{l^*E_1}\psi^{(\lambda)}$:*

$$\begin{aligned}\left(E_1^{(\lambda+1)}, \ldots, E_n^{(\lambda+1)}\right) &= \left(E_2^{(\lambda)}, \ldots, E_n^{(\lambda)}, E_1^{(\lambda)}, \otimes L^{\otimes(r-1)}\right) \\ \psi^{(\lambda+1)} &= \mathbb{R}_{l^*E_1}\psi^{(\lambda)}\end{aligned} \tag{10.3.34}$$

or sending $E_n^{(\lambda)}$ to the beginning, tensor it with $L^{\otimes(-r+1)}$ and substitute $\psi^{(\lambda)}$ with $\mathbb{L}_{l^*(E_n \otimes L^{\otimes(-r+1)})}\psi^{(\lambda)}$:

$$\begin{aligned}\left(E_1^{(\lambda+1)}, \ldots, E_n^{(\lambda+1)}\right) &\longrightarrow \left(E_1^{(\lambda)} \otimes L^{\otimes(-r+1)}, E_2^{(\lambda)}, \ldots, E_{n-1}^{(\lambda)}\right) \\ \psi^{(\lambda+1)} &\longrightarrow \mathbb{L}_{l^*(E_n \otimes L^{\otimes(-r+1)})}\psi^{(\lambda)}\end{aligned} \tag{10.3.35}$$

O3 *For any $k \in \mathbb{Z}$, substituting $E_i^{(\lambda)}$ with $E_i^{(\lambda)} \otimes L^{\otimes k}$ for every i, and substituting $\psi^{(\lambda)}$ with $\psi^{(\lambda)}(-) \otimes L^{\otimes k}$:*

$$\begin{gathered}\left(E_1^{(\lambda+1)}, \dots, E_n^{(\lambda+1)}\right) = \left(E_1^{(\lambda)} \otimes L^{\otimes k}, \dots, E_n^{(\lambda)} \otimes L^{\otimes k}\right) \\ \psi^{(\lambda+1)} = \psi^{(\lambda)}(-) \otimes L^{\otimes k}\end{gathered} \tag{10.3.36}$$

- *An autoequivalence of $D^b(M)$ given by a sequence of mutations and twists on the semiorthogonal decompositions 10.3.25 which acts in the following way:*

$$\langle \theta_1 D^b(Y_1), l^* E_1, \dots, l^* E_n \rangle \longrightarrow \langle \psi D^b(Y_1), l^* F_1, \dots, l^* F_n \rangle \tag{10.3.37}$$

Lemma 10.3.7. *Let $D^b(G/P) = \langle W_1, \dots, W_n, \rangle$ be a full exceptional collection. Then, in the setting of Notation 10.3.6, for $1 \leq i \leq N-1$ one has:*

$$D^b(\mathcal{Z}) = \langle [W_1], \dots, [W_{i-1}], [\mathbb{L}_{W_i} W_{i+1}], [W_i], [W_{i+2}], \dots, [W_N] \rangle \tag{10.3.38}$$

Moreover, the following holds:

$$\mathbb{L}_{[W_i]}[W_{i+1}] \simeq [\mathbb{L}_{W_i} W_{i+1}] \tag{10.3.39}$$

Proof. By Theorem 3.3.2 one has a semiorthogonal decomposition $D^b(\mathcal{Z}) = \langle [W_1], \dots, [W_N] \rangle$, then applying (Kuz10, Corollary 2.9) one finds:

$$D^b(\mathcal{Z}) = \langle [W_1], \dots, [W_{i-1}], \mathbb{L}_{[W_i]}[W_{i+1}], [W_i], [W_{i+2}], \dots, [W_N] \rangle \tag{10.3.40}$$

On the other hand, for $1 \leq i \leq N-1$ one has:

$$D^b(G/P) = \langle W_1, \ldots, W_{i-1}, \mathbb{L}_{W_i} W_{i+1}, W_i, W_{i+2}, \ldots, W_m \rangle \qquad (10.3.41)$$

and by applying again Theorem 3.3.2 to the collection 10.3.41 one finds:

$$D^b(\mathcal{Z}) = \langle [W_1], \ldots, [W_{i-1}], [\mathbb{L}_{W_i} W_{i+1}], [W_i], [W_{i+2}], \ldots, [W_N] \rangle \quad (10.3.42)$$

By comparison with Equation 10.3.40 we see that both $[\mathbb{L}_{W_i} W_{i+1}]$ and $\mathbb{L}_{[W_i]}[W_{i+1}]$ are equivalent to the subcategory ${}^{\perp}\langle [W_1], \ldots, [W_{i-1}] \rangle \cap \langle [W_i], [W_{i+2}], \ldots, [W_N] \rangle^{\perp}$, hence they are equivalent. □

Lemma 10.3.8. *In the language of Notation 10.3.6, consider a semiorthogonal decomposition $D^b(M) = \langle \phi_1 D^b(Y_1), l^* W_1, \ldots, l^* W_n \rangle$ where, for $1 \leq j \leq n$, W_j is a homogeneous vector bundle on G/P. Assume that W_i is L-semiorthogonal to W_{i+1} for some positive $i < n$. Then the following holds:*

- $\mathbb{L}_{W_i} W_{i+1}$ *commutes with* l^*
- $\mathbb{L}_{[W_i]}[W_{i+1}]$ *commutes with* ι^*
- *one has a semiorthogonal decomposition:*

$$\begin{aligned} D^b(\mathcal{M}) = &\langle \phi_1 D^b(X_1), \iota^*[W_1], \ldots\ldots, \iota^*[W_{i-1}], \\ &\iota^*[\mathbb{L}_{W_i} W_{i+1}], \iota^*[W_i], \iota^*[W_{i+2}], \ldots, \iota^*[W_n] \rangle \end{aligned} \qquad (10.3.43)$$

Proof. Let us first recall that by Theorem 3.3.2 and Theorem 3.3.5 one has $D^b(\mathcal{M}) = \langle \phi_1(Y_1), \iota^*[W_1], \ldots, \iota^*[W_n] \rangle$. We now prove commutativity. By Lemma 10.3.7, one has $[\mathbb{L}_{W_i} W_{i+1}] = \mathbb{L}_{[W_i]}[W_{i+1}]$. By Lemma 10.3.5, since W_i is L-semiorthogonal to W_{i+1} it follows that

$\mathcal{W}_i$ is $\mathcal{L}$-semiorthogonal to $\mathcal{W}_{i+1}$. Then, by Lemma 10.3.4 applied to $G/P \longrightarrow \{pt\}$ one finds that $\mathbb{L}_{W_i} W_{i+1}$ commutes with l^*, while by applying the same lemma to $\mathcal{Z} \longrightarrow B$ it follows that $\mathbb{L}_{[W_i]}[W_{i+1}]$ commutes with ι^*. By the latter we get:

$$\mathbb{L}_{\iota^*[W_i]}\iota^*[W_{i+1}] \simeq \iota^*\mathbb{L}_{[W_i]}[W_{i+1}] = \iota^*[\mathbb{L}_{W_i} W_{i+1}] \tag{10.3.44}$$

By (Kuz10, Corollary 2.9) one has:

$$\begin{aligned} D^b(\mathcal{M}) =& \langle \phi_1 D^b(X_1), \iota^*[W_1], \dots\dots\dots, \iota^*[W_{i-1}], \\ & \mathbb{L}_{\iota^*[W_i]}\iota^*[W_{i+1}], \iota^*[W_i], \iota^*[W_{i+2}], \dots, \iota^*[W_n]\rangle \end{aligned} \tag{10.3.45}$$

and substituting 10.3.44 in the decomposition 10.3.43 completes the proof.

□

Proposition 10.3.9. *In the language of Notation 10.3.6, assume there is a semiorthogonal decomposition $D^b(M) = \langle \phi_1 D^b(Y_1), l^*W_1, \dots, l^*W_n\rangle$ where every W_j is a homogeneous vector bundle on G/P. Then one has:*

$$D^b(\mathcal{M}) = \langle \mathbb{L}_{\iota^*[W_n \otimes L^{\otimes(-r+1)}]}\phi_1 D^b(X_1), \iota^*[W_n \otimes L^{\otimes(-r+1)}], \iota^*[W_1], \dots, \iota^*[W_{n-1}]\rangle. \tag{10.3.46}$$

Proof. Since $\mathcal{M}$ is a smooth projective variety, by the Serre functor, there is the following semiorthogonal decomposition:

$$D^b(\mathcal{M}) = \langle \mathbb{L}_{\iota^*[W_n]\otimes\omega_{\mathcal{M}}}\phi_1 D^b(X_1), \iota^*[W_n] \otimes \omega_{\mathcal{M}}, \iota^*[W_1], \dots, \iota^*[W_{n-1}]\rangle. \tag{10.3.47}$$

One has (Ful98, Example 3.2.11):

$$\omega_{\mathcal{Z}} \simeq p_i^*\omega_{\mathcal{Z}_i} \otimes p_i^* \det \mathcal{E}_i \otimes \mathcal{L}^{\otimes(-r)} \tag{10.3.48}$$

but since $\mathcal{Z}$ is a roof bundle, $(\mathcal{Z}|_{r_i^{-1}(b)}, \mathcal{E}_i|_{r_i^{-1}(b)})$ is a Mukai pair for every $b \in B$, which implies that $\omega_{\mathcal{Z}_i} \otimes \det \mathcal{E}_i \simeq r_i^* T$, where T is a line bundle on B. Then, by plugging this into 10.3.48 we get $\omega_{\mathcal{Z}} \simeq \mathcal{L}^{\otimes(-r)} \otimes \pi^* T$. Due to the following normal bundle sequence:

$$0 \longrightarrow T_{\mathcal{M}} \longrightarrow \iota^* T_{\mathcal{Z}} \longrightarrow \mathcal{L} \longrightarrow 0 \tag{10.3.49}$$

one has $\omega_{\mathcal{M}} \simeq \iota^* \omega_{\mathcal{Z}} \otimes \iota^* \mathcal{L}^\vee \simeq \iota^* \mathcal{L}^{\otimes(-r+1)} \otimes \iota^* \pi^* T$. Then, the proof is completed by the following computation:

$$\iota^*[W_n] \otimes \omega_{\mathcal{M}} = \iota^*[W_n] \otimes \iota^* \mathcal{L}^{\otimes(-r+1)} = \iota^*[W_n \otimes L^{\otimes(-r+1)}]. \tag{10.3.50}$$

and by plugging it in the decomposition 10.3.47. $\square$

Let us gather here the assumptions for the main theorem of this chapter.

Assumption 10.3.10. *The data of Notation 10.3.6 fulfill the following requirements:*

A1 *$\mathcal{L}$ is basepoint-free and the restriction map*

$$H^0(\mathcal{Z}, \mathcal{L}) \longrightarrow H^0(\pi^{-1}(b), \mathcal{L}|_{\pi^{-1}(b)}) \tag{10.3.51}$$

is surjective for every $b \in B$

A2 *The autoequivalence of $D^b(M)$ described in Notation 10.3.6 acts by a composition of the following operations on the first collection of Equation 10.3.25:*

- *Mutations of pairs of exceptional objects $\mathbb{L}_{l^*E} l^* F$ where E, F satisfy the semiorthogonality condition $\operatorname{Ext}^\bullet_{G/P}(F, E) = 0$*

and E is L-semiorthogonal to F (in short, $\mathbb{L}_{l^*E} l^*F$ satisfies Condition (†) as defined in Chapter 9)

- *Overall twists by a power of L.*
- *Applying the Serre functor of $\mathcal{S}_M$ sending the last exceptional object to the beginning of the semiorthogonal decompositions, or applying the inverse functor $\mathcal{S}_M^{-1}$.*
- *Applying the mutation $\mathbb{L}_{l^*E}$ or $\mathbb{R}_{l^*E}$ to the subcategory $\theta_i D^b(Y_i)$, where l^*E is an exceptional object in the right (respectively left) semiorthogonal complement of $\theta_i D^b(Y_i)$.*

A3 *The sequence of operations $\xi = \xi^{(R)} \dots \xi^{(1)}$ acts on $(\boldsymbol{E}, \theta_1)$ in the following way:*

$$\xi : (\boldsymbol{E}, \theta_1) \longmapsto (\boldsymbol{F}, \psi) \tag{10.3.52}$$

where $(\boldsymbol{E}^{(1)}, \psi^{(1)}) = (\boldsymbol{E}, \theta_1)$ and $(\boldsymbol{E}^{(R+1)}, \psi^{(R+1)}) = (\boldsymbol{F}, \psi)$.

The condition A1 is needed to ensure smoothness of the general sections of $\mathcal{L}$, and the fact that zero loci of pushforwards of such sections have the property of being Calabi–Yau fibrations. On the other hand, assumption A2 is needed to construct the mutations in $D^b(\mathcal{M})$. The last assumption A3 is needed to ensure that such mutations really yield an equivalence $D^b(X_1) \simeq D^b(X_2)$.

Definition 10.3.11. *In the language of Notation 10.3.6, we say that a pair $(\mathcal{Z}, \mathcal{L})$ of a roof bundle $\mathcal{Z}$ of type G/P together with the line bundle $\mathcal{L}$ satisfies Assumption 10.3.10 if $\mathcal{L}$ satisfies Assumption A1 and there exist two full exceptional collections $G/P = \langle E_1, \dots E_N \rangle = \langle F_1 \dots F_N \rangle$ and a section $\sigma \in H^0(G/P, L)$ as required in Notation 10.3.6, which are* *compatible with Assumptions A2 and A3.*

Theorem 10.3.12. *Let $(\mathcal{Z}, \mathcal{L})$ satisfy Assumption 10.3.10. Then a general section of $\mathcal{L}$ induces a derived equivalence of Calabi–Yau fibrations $\Phi : D^b(X_1) \longrightarrow D^b(X_2)$.*

Proof. Consider two full exceptional collections $G/P = \langle E_1, \ldots E_N \rangle = \langle F_1 \ldots F_N \rangle$ and a general section $\sigma \in H^0(G/P, L)$ with zero locus M, compatible with Assumptions A2 and A3. Then, by Assumption A2 one has a derived equivalence:

$$D^b(M) = \langle \theta_1 D^b(Y_1), l^* E_1, \ldots, l^* E_n \rangle \tag{10.3.53}$$

$$\longrightarrow \langle \psi D^b(Y_1), l^* F_1, \ldots, l^* F_n \rangle \tag{10.3.54}$$

which consists in applying a sequence of operations to the first semiorthogonal decomposition, which only include mutations of L-semiorthogonal exceptional pairs, the Serre functor of M, overall twists by a line bundle and mutations of $\theta_1 D^b(Y_1)$ through the admissible subcategory generated by an exceptional object.

Consider the sequence of operations $\xi = \xi^{(R)} \cdots \xi^{(1)}$ defined in Notation 10.3.6. By Assumption A3 one has:

$$\xi : (\boldsymbol{E}, \theta_1) \longmapsto (\boldsymbol{F}, \psi) \tag{10.3.55}$$

These operations, by A2, are in one-to-one correspondence with the mutations used to transform the decomposition 10.3.53 into 10.3.54 (the case of mutations of pairs is treated in Lemma 10.3.8). In fact, for $1 \leq \lambda \leq R + 1$ one has a semiorthogonal decomposition:

$$D^b(M) = \langle \psi^{(\lambda)} D^b(Y_1), l^* E_1^{(\lambda)}, \ldots, l^* E_n^{(\lambda)} \rangle. \tag{10.3.56}$$

Every operation $\xi^{(\lambda)}$ commutes also with the mapping $W \longmapsto [W]$: for the cases of Operations O2 and O3 this follows by the fact that the product of homogeneous vector bundles associated to representations Γ, Γ' is the homogeneous vector bundle associated to the representation $\Gamma \otimes \Gamma'$, while the case of type O1 follows from Lemma 10.3.7.

Therefore, we can define $[\boldsymbol{E}] = \{[E_1], \dots, [E_n]\}$, $[\boldsymbol{F}] = \{[F_1], \dots, [F_n]\}$ and construct a sequence of operations $[\xi] = [\xi^{(R)}] \cdots [\xi^{(1)}]$ as follows:

$$[\xi^{(\lambda)}] : ([\boldsymbol{E}^{(\lambda)}], \Phi^{(\lambda)}) \longmapsto ([\boldsymbol{E}^{(\lambda+1)}], \Phi^{(\lambda+1)}) \tag{10.3.57}$$

where $([\boldsymbol{E}^{(1)}], \Phi^{(1)}) = ([\boldsymbol{E}], \phi_1)$ and $([\boldsymbol{E}^{(R+1)}], \Phi^{(R+1)}) = ([\boldsymbol{F}], \Phi)$. The functor $\Phi^{(\lambda)}$ is defined by formally substituting $\iota^*[E_i]$ in place of $l^* E_i$ in the definition of $\psi^{(\lambda)}$.

Fix a general section $\Sigma \in H^0(\mathcal{Z}, \mathcal{L})$ with zero locus $\mathcal{M}$. Hereafter we show that we can associate to the sequence of mutations on the collection 10.3.53 a sequence of mutations on the decomposition $D^b(\mathcal{M}) = \langle \phi_1 D^b(X_1), \iota^*[E_1], \dots, \iota^*[E_n] \rangle$ defined through the operations $[\xi^{(\lambda)}]$, thus obtaining for every λ:

$$D^b(\mathcal{M}) = \langle \Phi^{(\lambda)} D^b(X_1), \iota^*[E_1^{(\lambda)}], \dots, \iota^*[E_n^{(\lambda)}] \rangle. \tag{10.3.58}$$

To prove our claim, let us consider each of the allowed kinds of mutations on 10.3.53, and describe the associated mutation on 10.3.58.

- Every time a left mutation of pairs

$$\langle \dots, l^* E_i^{(\lambda)}, l^* E_{i+1}^{(\lambda)}, \dots \rangle \longrightarrow \langle \dots, \mathbb{L}_{l^* E_i^{(\lambda)}} l^* E_{i+1}^{(\lambda)}, l^* E_i^{(\lambda)}, \dots \rangle \tag{10.3.59}$$

is performed in 10.3.56, we do the operation

$$\begin{aligned}
&\langle \Phi^{(\lambda)} D^b(X_1), \ldots, \iota^*[E_i^{(\lambda)}], \iota^*[E_{i+1}^{(\lambda)}], \ldots \rangle \\
&\longrightarrow \langle \Phi^{(\lambda)} D^b(X_1), \ldots, \iota^*[\mathbb{L}_{E_i^{(\lambda)}} E_{i+1}^{(\lambda)}], [E_i^{(\lambda)}], \ldots \rangle \\
&=: \langle \Phi^{(\lambda+1)} D^b(X_1), \iota^*[E_1^{(\lambda+1)}], \ldots, \iota^*[E_n^{(\lambda+1)}] \rangle
\end{aligned} \tag{10.3.60}$$

in 10.3.58.

We obtain a semiorthogonal decomposition because of the following argument: by Assumption A2, $E_i^{(\lambda)}$ is L-semiorthogonal to $E_{i+1}^{(\lambda)}$, thus by Lemma 10.3.8, $\mathbb{L}_{E_i^{(\lambda)}} E_{i+1}^{(\lambda)}$ commutes with l^* and $\mathbb{L}_{[E_i^{(\lambda)}]}[E_{i+1}^{(\lambda)}]$ commutes with ι^*. Finally, since $\mathbb{L}_{[E_i^{(\lambda)}]}[E_{i+1}^{(\lambda)}] = [\mathbb{L}_{E_i^{(\lambda)}} E_{i+1}^{(\lambda)}]$ by Lemma 10.3.7, we see that the operation described in Equation 10.3.60 is simply the left mutation of $\iota^*[E_{i+1}^{(\lambda)}]$ through $\iota^*[E_i^{(\lambda)}]$. An analogous argument works for right mutations.

- Every time the Serre functor is applied to Equation 10.3.56:

$$\begin{aligned}
&\langle \psi^{(\lambda)} D^b(Y_1), l^* E_1^{(\lambda)}, \ldots, l^* E_n^{(\lambda)}, \rangle \\
&\longrightarrow \langle \mathbb{L}_{l^* E_n^{(\lambda)} \otimes L^{-r+1}} \psi^{(\lambda)} D^b(Y_1), l^* E_n^{(\lambda)} \otimes L^{-r+1}, l^* E_1^{(\lambda)}, \ldots, l^* E_{n-1}^{(\lambda)} \rangle
\end{aligned} \tag{10.3.61}$$

perform the following operation on Equation 10.3.58:

$$\begin{aligned}
&\langle \Phi^{(\lambda)} D^b(X_1), \iota^*[E_1^{(\lambda)}], \ldots, \iota^*[E_n^{(\lambda)}], \rangle \\
&\longrightarrow \langle \mathbb{L}_{[\iota^* E_n^{(\lambda)} \otimes L^{-r+1}]} \Phi^{(\lambda)} D^b(X_1), [\iota^* E_n^{(\lambda)} \otimes L^{-r+1}], \\
&\qquad \iota^*[E_1^{(\lambda)}], \ldots, \iota^*[E_{n-1}^{(\lambda)}] \rangle \\
&=: \langle \Phi^{(\lambda+1)} D^b(X_1), \iota^*[E_1^{(\lambda+1)}], \ldots, \iota^*[E_n^{(\lambda+1)}] \rangle
\end{aligned} \tag{10.3.62}$$

In fact, by Proposition 10.3.9 the resulting collection above is the one obtained by applying the Serre functor of $\mathcal{M}$ to $\iota^*[E_1^{(\lambda)}]$ and sending the subcategory equivalent to $D^b(X_1)$ to the beginning of the collection. The same holds for the inverse Serre functor.

- Whenever Equation 10.3.56 is twisted by $L^{\otimes k}$ for some $k \in \mathbb{Z}$, perform the following operation on Equation 10.3.58:

$$\begin{aligned} &\langle \Phi^{(\lambda)} D^b(X_1), \iota^*[E_1^{(\lambda)}], \dots, \iota^*[E_n^{(\lambda)}], \rangle \\ &\longrightarrow \langle \Phi^{(\lambda)} D^b(X_1) \otimes \mathcal{L}^{\otimes k}, \iota^*[E_1^{(\lambda)} \otimes L^{\otimes k}], \dots, \iota^*[E_n^{(\lambda)} \otimes L^{\otimes k}], \rangle \\ &\quad =: \langle \Phi^{(\lambda+1)} D^b(X_1), \iota^*[E_1^{(\lambda+1)}], \dots, \iota^*[E_n^{(\lambda+1)}] \rangle \end{aligned} \tag{10.3.63}$$

In this way, we showed that for every λ there is the following semiorthogonal decomposition:

$$D^b(\mathcal{M}) = \langle \Phi^{(\lambda)} D^b(X_1), \iota^*[E_1^{(\lambda)}], \dots, \iota^*[E_n^{(\lambda)}] \rangle. \tag{10.3.64}$$

In particular, for $\lambda = R + 1$ we obtain:

$$D^b(\mathcal{M}) = \langle \Phi D^b(X_1), \iota^*[F_1], \dots, \iota^*[F_n] \rangle \tag{10.3.65}$$

On the other hand, starting from the full exceptional collection $G/P_2 = \langle K_1, \dots, K_m \rangle$, by (Sam06, Theorem 3.1) and (Orl03, Proposition 2.10) one has

$$D^b(\mathcal{M}) = \langle \phi_2 D^b(X_2), \iota^*[F_1], \dots, \iota^*[F_n] \rangle \tag{10.3.66}$$

The proof is completed by comparing the decomposition 10.3.65 with 10.3.66. □

Remark 10.3.13. Note that Theorem 10.3.12 holds for any nonzero $\Sigma \in H^0(\mathcal{Z}, \mathcal{L})$ with smooth zero locus, if the pair $(\mathcal{Z}, \mathcal{L})$ satisfies Assumption 10.3.10. In fact, by (Orl03 Proposition 2.10) every nonzero section with smooth zero locus $\mathcal{M}$ admits the semiorthogonal decompositions 10.3.28.

As we will show in Lemma 10.3.14, Theorem 10.3.12 can be immediately applied to all cases of roofs where a sequence of mutations realizing a derived equivalence of a Calabi–Yau pair is known, provided that $\mathcal{L}$ satisfies Assumption A1.

Lemma 10.3.14. *Derived equivalences of Calabi–Yau pairs associated to roofs of type A_n^M, $A_n \times A_n$, A_4^G, C_2 and G_2 satisfy Assumptions A2 and A3.*

Proof. Let G/P be a roof of the types listed above. Then, in the language of Notation 10.3.6, in all these cases there is a sequence of mutations of the semiorthogonal decomposition 10.1.3: such mutations are either applications of the Serre functor, overall twists by a power of L, mutations of a subcategory equivalent to $D^b(Y_1)$ through an exceptional object, or mutations of pairs of exceptional objects. Hence, the proof reduces to ensure that every time a mutation of pairs $\mathbb{L}_{l^*E} l^*F$ or $\mathbb{R}_{l^*F} l^*E$ is performed, such mutation satisfies Condition (†). But this property has been checked case by case with Lemma 9.3.3 for A_n^M, Lemma 9.4.1 for $A_n \times A_n$, Lemma 9.2.1 for C_2 and Lemma 9.5.7 for A_4^G, while the case of G_2 is treated in Corollary 9.6.1. □

Even if a general proof is lacking, for all roof bundles where a proof of derived equivalence based on mutations of the associated Calabi–Yau pair is known, such mutations satisfy Assumptions A2 and A3. Therefore, in light of (KR20, Conjecture 2.6), we formulate the following:

Conjecture 10.3.15. *Let G/P be a homogeneous roof, and $\mathcal{Z}$ a homogeneous roof bundle of type G/P with projective bundle structures $p_i : \mathcal{Z} \longrightarrow \mathcal{Z}_i$ for $i \in \{1; 2\}$. Given a general section $\Sigma \in H^0(\mathcal{Z}, \mathcal{L})$, the*

Calabi–Yau fibrations $X_i := Z(p_{i*}\Sigma)$ are derived equivalent.

We summarize all the evidence we have in support of Conjecture 10.3.15 in the form of a corollary for Theorem 10.3.12 and Lemma 10.3.14.

Corollary 10.3.16. *Let $\mathcal{Z}$ be a roof bundle of type $A_n \times A_n$, A_n^M, A_4^G, C_2 or G_2 and let $\mathcal{L}$ satisfy Assumption A1. Then, given a general section Σ of $\mathcal{L}$, the associated pair of Calabi–Yau fibrations is derived equivalent.*

Proof. By Lemma 10.3.14, a Calabi–Yau pair associated to a roof of the types listed above is derived equivalent by means of a sequence of mutations satisfying Assumptions $A2$ and $A3$. Thus, by requiring that $\mathcal{L}$ satisfies Assumption $A1$, the claim is proven by Theorem 10.3.12. □

10.4 Universal hyperplane sections

In this section we will present a different version of the argument above, which allows to prove derived equivalence of the *universal* pair of Calabi–Yau fibrations under the same assumptions. More precisely, we will construct a sequence of mutations providing an equivalence of categories for zero loci of pushforwards of the universal hyperplane section of $\mathcal{Z}$ with respect to the hyperplane bundle $\mathcal{L}$ of Assumption 10.3.10. For the general definitions and properties of universal hyperplane sections we refer to Section 3.4 and the main source therein, which is (Kuz07), while we develop here the notation which will be used in the rest of the section.

Let $\mathcal{Z}$ be a homogeneous roof bundle, fix $\mathcal{H} = \mathbb{P}(H^0(\mathcal{Z},\mathcal{L}))$ where

$\mathcal{L}$ satisfies the hypotheses of Assumption 10.3.10. The universal hyperplane section of $\mathcal{Z}$ with respect to the map $\phi_{\mathcal{L}} : \mathcal{Z} \longrightarrow \mathcal{H}$ (which exists because $\mathcal{L}$ is basepoint-free) is the following variety:

$$\widehat{\mathcal{M}} := \{(x, \Sigma) \in \mathcal{Z} \times \mathcal{H} \;:\; \Sigma(x) = 0\}. \tag{10.4.1}$$

In fact, the incidence quadric in $\mathcal{H} \times \mathcal{H}^\vee$ is exactly $Q = \{(x, y) \in \mathcal{H} \times \mathcal{H}^\vee : y(x) = 0\}$, therefore one has $\widehat{\mathcal{M}} = \mathcal{Z} \times_{\mathcal{H}} Q$ as in Definition 3.4.1.

As a shorthand notation, let us call $\widehat{\mathcal{Z}} := \mathcal{Z} \times \mathcal{H}$ and introduce the line bundle $\widehat{\mathcal{L}} := \mathcal{L} \boxtimes \mathcal{O}_{\mathcal{H}}(1)$. A useful characterization for $\widehat{\mathcal{M}}$ is given as a zero locus $\widehat{\mathcal{M}} = Z(S)$, where $S \in H^0(\widehat{\mathcal{Z}}, \widehat{\mathcal{L}})$ is the section defined by $S(x, \Sigma) = \Sigma(x)$. By the fact that $\mathcal{L}$ is basepoint-free we deduce that $\widehat{\mathcal{M}}$ is smooth, since $Z(S)$ is isomorphic to the zero locus of a general section of $\widehat{\mathcal{L}}$. We can write the following Koszul resolution:

$$0 \longrightarrow \widehat{\mathcal{L}}^\vee \longrightarrow \mathcal{O}_{\widehat{\mathcal{Z}}} \longrightarrow \widehat{i}_*\mathcal{O}_{\widehat{\mathcal{M}}} \longrightarrow 0 \tag{10.4.2}$$

where $\widehat{i} : \widehat{\mathcal{M}} \longrightarrow \widehat{\mathcal{Z}}$. We can establish a construction like Diagram 10.1.1 replacing each variety with its "universal" counterpart. More precisely, we see that $\widetilde{\mathcal{Z}}$ is the projectivization of the pullbacks $\widehat{\mathcal{E}}_i$ to $\widehat{\mathcal{Z}}_i$ of $\mathcal{E}_i$, therefore, if we define $\widehat{p}_i := (p_i, Id)$, we have $\widehat{p}_{i*}\widehat{\mathcal{L}} = \widehat{\mathcal{E}}_i$.

Definition 10.4.1. *Let $\mathcal{Z}$ be a homogeneous roof bundle We call* universal Calabi–Yau fibrations of $\mathcal{Z}$ *the varieties:*

$$\widehat{X}_i := \{(x, \Sigma) \in \widehat{\mathcal{Z}}_i \;:\; p_{i*}\Sigma(x) = 0\}. \tag{10.4.3}$$

for $i \in \{1; 2\}$.

Again, we can characterize $\widehat{X}_i$ as the zero locus $\widehat{X}_i = Z(\widehat{p}_{i*}S) \subset \widehat{\mathcal{Z}}_i$. We can summarize this setting with the following diagram:

$$
\begin{tikzcd}
 & \widehat{\mathcal{T}}_1 \arrow[r, hook, "\widehat{m}_1"] \arrow[ddl, "\widehat{\mu}_1"'] & \widehat{\mathcal{M}} \arrow[d, hook, "\widehat{i}"'] & \widehat{\mathcal{T}}_2 \arrow[l, hook', "\widehat{m}_2"'] \arrow[ddr, "\widehat{\mu}_2"] & \\
 & & \widehat{\mathcal{Z}} \arrow[dl, "\widehat{p}_1"'] \arrow[dr, "\widehat{p}_2"] & & \\
\widehat{X}_1 \arrow[r, hook, "\widehat{u}_1"] & \widehat{\mathcal{Z}}_1 & & \widehat{\mathcal{Z}}_2 & \widehat{X}_2 \arrow[l, hook', "\widehat{u}_2"']
\end{tikzcd}
\tag{10.4.4}
$$

Note that there is a projection $\widehat{\pi} : \mathcal{Z} \times \mathcal{H} \longrightarrow B$ defined as the composition of π with the projection of $\mathcal{Z} \times \mathcal{H}$ to the first factor. For every $b \in B$ one has $\widehat{\pi}^{-1}(b) \simeq G/P \times \mathcal{H} := \widehat{G/P}$. Hence the same construction as above, restricted to a single fiber of $\widehat{\pi}$, gives rise to the following diagram:

$$
\begin{tikzcd}
 & \widehat{T}_1 \arrow[r, hook, "\widehat{k}_1"] \arrow[ddl, "\widehat{\nu}_1"'] & \widehat{M}_b \arrow[d, hook, "\widehat{l}"'] & \widehat{T}_2 \arrow[l, hook', "\widehat{k}_2"'] \arrow[ddr, "\widehat{\nu}_2"] & \\
 & & \widehat{G/P} \arrow[dl, "\widehat{h}_1"'] \arrow[dr, "\widehat{h}_2"] & & \\
\widehat{Y}_{1b} \arrow[r, hook, "\widehat{t}_1"] & \widehat{G/P_1} & & \widehat{G/P_2} & \widehat{Y}_{2b} \arrow[l, hook', "\widehat{t}_2"']
\end{tikzcd}
\tag{10.4.5}
$$

where the variety $\widehat{M}_b$ can be described for every $b \in B$ by the following fibered square:

$$
\begin{tikzcd}
\widehat{M}_b = \widehat{\mathcal{M}} \times_B \{b\} \arrow[r, "\tau"] \arrow[d] & \widehat{\mathcal{M}} \arrow[d, "\widehat{\pi}\circ\widehat{i}"] \\
\{b\} \arrow[r, hook, "f_b"] & B
\end{tikzcd}
\tag{10.4.6}
$$

Moreover, we have the square:

$$\begin{tikzcd} \widehat{G/P} \simeq \widehat{\mathcal{Z}} \times_B \{b\} \arrow[r, "\rho"] \arrow[d, "\widehat{\pi}_b"] & \widehat{\mathcal{Z}} \arrow[d, "\widehat{\pi}"] \\ \{b\} \arrow[r, hook, "f_b"] & B \end{tikzcd} \tag{10.4.7}$$

The analogy of these universal constructions with the ones presented in Diagram 10.1.1 and 9.1.1 extends to derived categories as well:

Lemma 10.4.2. *Let $\widehat{\mathcal{M}}$ be the universal hyperplane section of a homogeneous roof bundle of type G/P. Then, there exists the following semiorthogonal decompositions for $i \in \{1; 2\}$:*

$$D^b(\widehat{\mathcal{M}}) = \langle \widehat{p}_i|^*_{\widehat{\mathcal{M}}} D^b(\widehat{\mathcal{Z}}_i), \cdots, \widehat{p}_i|^*_{\widehat{\mathcal{M}}} D^b(\widehat{\mathcal{Z}}_i) \otimes \widehat{\mathcal{L}}^{\otimes(r-2)}, \widehat{m}_{i*}\widehat{\mu}_i^* D^b(\widehat{X}_i)\rangle. \tag{10.4.8}$$

Moreover, given $b \in B$ such that $\widehat{M}_b$ is smooth, there exist the following semiorthogonal decompositions for $i \in \{1; 2\}$

$$D^b(\widehat{M}_b) = \langle \widehat{h}_i|^*_{\widehat{M}_b} D^b(\widehat{G/P_i}), \cdots, \widehat{h}_i|^*_{\widehat{M}_b} D^b(\widehat{G/P_i}) \otimes \widehat{L}^{\otimes(r-2)}, \widehat{k}_{i*}\widehat{\nu}_i^* D^b(\widehat{Y}_i)\rangle. \tag{10.4.9}$$

Proof. We have the following diagram:

$$\begin{tikzcd} \widehat{\mathcal{T}}_i \arrow[r, hook, "\widehat{m}_i"] \arrow[d, "\widehat{\mu}_i"] & \widehat{\mathcal{M}} \arrow[r, hook, "\widehat{i}"] \arrow[d, "\widehat{p}_i|_{\widehat{\mathcal{M}}}"] & \mathbb{P}(\widehat{\mathcal{E}}_i) \simeq \widehat{\mathcal{Z}} \arrow[dl, "\widehat{p}_i"] \\ \widehat{X}_i \arrow[r, hook, "\widehat{u}_i"] & \widehat{\mathcal{Z}}_i & \end{tikzcd} \tag{10.4.10}$$

which on the fiber $\widehat{\pi}^{-1}(b)$ restricts to:

$$
\begin{tikzcd}
\widehat{T}_i \arrow[r, hook, "\widehat{k}_i"] \arrow[d, "\widehat{v}_i"] & \widehat{M}_b \arrow[r, hook, "\widehat{l}"] \arrow[d, "\widehat{h}_i|_{\widehat{\mathcal{M}}}"] & \widehat{\mathbb{P}}(\widehat{E}_i) \simeq \widehat{G/P} \arrow[dl, "\widehat{h}_i"] \\
\widehat{X}_i \arrow[r, hook, "\widehat{t}_i"] & \widehat{G/P_i} &
\end{tikzcd}
\tag{10.4.11}
$$

The proof is simply an application of Cayley trick (Orl03) to Diagrams 10.4.10 and 10.4.11. □

Theorem 10.4.3. *Let the data $(\mathcal{Z}, \mathcal{L})$ satisfy Assumption 10.3.10. Then, there exists an autoequivalence $\Phi : D^b(\widehat{\mathcal{M}}) \longrightarrow D^b(\widehat{\mathcal{M}})$ which provides an equivalence of categories $D^b(\widehat{X}_1) \longrightarrow D^b(\widehat{X}_2)$, where $(\widehat{X}_1, \widehat{X}_2)$ is the pair of universal Calabi–Yau fibrations in the sense of Definition 10.4.1.*

Proof. Let us observe that $\widehat{\mathcal{Z}}$ is itself a roof bundle of type G/P over the base $B \times \mathcal{H}$. Recall that $\widehat{\mathcal{M}}$ is the zero locus of a section of $\widehat{\mathcal{L}} = \mathcal{L} \boxtimes \mathcal{O}_{\mathcal{H}}(1)$. Then, if we choose $(b, \Sigma) \in B \times \mathcal{H}$ such that $M_{b,\Sigma} := \widehat{\mathcal{M}} \times_{B \times \mathcal{H}} \{(b, \Sigma)\} \subset G/P$ is smooth of expected codimension, the data of $M_{b,\Sigma}$ define a Calabi–Yau pair (Y_1, Y_2) associated to a roof of type G/P. Thus, applying Theorem 10.3.12 proves that the equivalence $D^b(Y_1) \simeq D^b(Y_2)$ induces a derived equivalence of $\widehat{X}_1$ and $\widehat{X}_2$. □

10.5 An explicit computation: mutations for a roof bundle of type A_4^G

Let us recall here the geometry of the roof bundle of type A_4^G over $\mathbb{P}(V_6)$, where V_6 is a vector space of dimension six. If we fix $\mathcal{V} =$

$T_{\mathbb{P}^5}(-1)$, we obtain $\mathcal{Z} = \mathcal{F}l(2,3,\mathcal{V}) \simeq F(1,3,4,V_6)$ and the associated Grassmann bundles are $\mathcal{G}r(2,\mathcal{V}) \simeq F(1,3,V_6)$ and $\mathcal{G}r(3,\mathcal{V}) \simeq F(1,4,V_6)$:

$$\begin{tikzcd}
 & & F(1,3,4,V_6) \arrow[dl, "p_1"'] \arrow[dd, "\pi"] \arrow[dr, "p_2"] & & \\
 & F(1,3,V_6) \arrow[dl, "\rho"'] \arrow[dr, "r_1"] & & F(1,4,V_6) \arrow[dl, "r_2"'] \arrow[dr, "\tau"] & \\
G(3,V_6) & & G(1,V_6) & & G(4,V_6)
\end{tikzcd} \tag{10.5.1}$$

Fix $\mathcal{L} = \mathcal{O}(1,1,1)$ where we call $\mathcal{O}(a,b,c) = \pi^*\mathcal{O}(a) \otimes p_1^*\rho^*\mathcal{O}(b) \otimes p_2^*\tau\mathcal{O}(c)$, call $\mathcal{M} = Z(\Sigma)$. In the notation of Diagram 10.1.1, by Cayley trick we have:

$$\begin{aligned} D^b(\mathcal{M}) \simeq& \langle p_1|_{\mathcal{M}}^* D^b\mathcal{G}r(2,\mathcal{V}), p_1|_{\mathcal{M}}^* D^b\mathcal{G}r(2,\mathcal{V}) \otimes \mathcal{O}(1,1,1), \phi_1 D^b(X_1)\rangle \\ \simeq& \langle p_2|_{\mathcal{M}}^* D^b\mathcal{G}r(3,\mathcal{V}), p_2|_{\mathcal{M}}^* D^b\mathcal{G}r(3,\mathcal{V}) \otimes \mathcal{O}(1,1,1), \phi_2 D^b(X_2)\rangle \end{aligned} \tag{10.5.2}$$

where $\phi_i = m_{i*} \circ \bar{p}_i^*$. To construct semiorthogonal decompositions for the Grassmann bundles we will use the following collections:

$$\begin{gathered} \{\mathcal{O}, \mathcal{P}^\vee, \mathcal{O}(0,1), \mathcal{P}^\vee(0,1), \mathcal{O}(0,2), \mathcal{P}^\vee(0,2), \mathcal{O}(0,3), \mathcal{P}^\vee(0,3), \mathcal{O}(0,4), \mathcal{P}^\vee(0,4)\} \\ \{\mathcal{O}, \mathcal{P}^\vee, \mathcal{O}(0,1), \widetilde{\mathcal{P}}^\vee(0,1), \mathcal{O}(0,2), \widetilde{\mathcal{P}}^\vee(0,2), \mathcal{O}(0,3), \widetilde{\mathcal{P}}^\vee(0,3), \mathcal{O}(0,4), \widetilde{\mathcal{P}}^\vee(0,4)\} \end{gathered} \tag{10.5.3}$$

where $\mathcal{P}_k$ is the tautological bundles of $\mathcal{G}r(k,\mathcal{V})$. In fact the objects of Equation 10.5.3 on every fiber over $\mathbb{P}^5$, reduce to the ones of the rectangular Lefschetz full exceptional collections of $G(2,5)$ and $G(3,5)$

provided in (Kuz08).

Following Notation 10.3.6 we construct the following blocks:

$$\begin{aligned}
[\mathcal{O}(a,b)] &:= \langle \mathcal{O}(0,a,b),\dots,\mathcal{O}(5,a,b)\rangle \\
[\mathcal{U}_2^\vee(a,b)] &:= \langle p_1^*\mathcal{P}_2^\vee(0,a,b),\dots,p_1^*\mathcal{P}_2^\vee(5,a,b)\rangle \\
[\mathcal{U}_3^\vee(a,b)] &:= \langle p_2^*\mathcal{P}_3^\vee(0,a,b),\dots,p_2^*\mathcal{P}_3^\vee(5,a,b)\rangle
\end{aligned} \tag{10.5.4}$$

from which, by the procedure described in Notation 10.3.6, we get:

$$\begin{aligned}
D^b(\mathcal{Z}) = \langle &[\mathcal{O}(0,0)],[\mathcal{U}_1^\vee(0,0)],\dots,[\mathcal{O}(4,0)],[\mathcal{U}_1^\vee(4,0)],\\
&[\mathcal{O}(1,1)],[\mathcal{U}_1^\vee(1,1)],\dots,[\mathcal{O}(5,1)],[\mathcal{U}_1^\vee(5,1)],\\
&[\mathcal{O}(2,2)],[\mathcal{U}_1^\vee(2,2)],\dots,[\mathcal{O}(6,2)],[\mathcal{U}_1^\vee(6,2)]\rangle\\
= \langle &[\mathcal{O}(0,0)],[\mathcal{U}_2^\vee(0,0)],\dots,[\mathcal{O}(0,4)],[\mathcal{U}_2^\vee(0,4)],\\
&[\mathcal{O}(1,1)],[\mathcal{U}_2^\vee(1,1)],\dots,[\mathcal{O}(1,5)],[\mathcal{U}_2^\vee(1,5)],\\
&[\mathcal{O}(2,2)],[\mathcal{U}_2^\vee(2,2)],\dots,[\mathcal{O}(2,6)],[\mathcal{U}_2^\vee(2,6)]\rangle
\end{aligned} \tag{10.5.5}$$

$$\begin{aligned}
D^b(\mathcal{M}) = \langle &\phi_1 D^b(X_1),\\
&\iota^*[\mathcal{O}(1,1)],\iota^*[\mathcal{U}_1^\vee(1,1)],\dots,\iota^*[\mathcal{O}(5,1)],\iota^*[\mathcal{U}_1^\vee(5,1)],\\
&\iota^*[\mathcal{O}(2,2)],\iota^*[\mathcal{U}_1^\vee(2,2)],\dots,\iota^*[\mathcal{O}(6,2)],\iota^*[\mathcal{U}_1^\vee(6,2)]\rangle\\
= \langle &\phi_2 D^b(X_2),\\
&\iota^*[\mathcal{O}(1,1)],\iota^*[\mathcal{U}_2^\vee(1,1)],\dots,\iota^*[\mathcal{O}(1,5)],\iota^*[\mathcal{U}_2^\vee(1,5)],\\
&\iota^*[\mathcal{O}(2,2)],\iota^*[\mathcal{U}_2^\vee(2,2)],\dots,\iota^*[\mathcal{O}(2,6)],\iota^*[\mathcal{U}_2^\vee(2,6)]\rangle
\end{aligned} \tag{10.5.6}$$

The derived equivalence $D^b(X_1) \simeq D^b(X_2)$ follows from Corollary 10.3.16, but let us compute some of the key mutations explicitly in this simple example. First, let us consider the mutations of the kind $\mathbb{L}_{\iota^*[(0,0)]}\iota^*[\mathcal{U}_2^\vee(0,0)] \simeq \iota^*[\mathcal{Q}_2(0,0)]$. By Equation 10.5.4, the pair $\langle \iota^*[(0,0)], \iota^*[\mathcal{U}_2^\vee(0,0)]\rangle$ can be written explicitly in the following way:

$$\langle \mathcal{O}_{\mathcal{M}}(0,0,0),\dots,\mathcal{O}_{\mathcal{M}}(5,0,0),\iota^* p_1^*\mathcal{P}_2^\vee(0,0,0),\dots,\iota^* p_1^*\mathcal{P}_2^\vee(5,0,0)\rangle. \tag{10.5.7}$$

One can compute $\mathrm{Ext}^\bullet_{\mathcal{M}}(\mathcal{O}_{\mathcal{M}}(k,0,0),\iota^* p_1^*\mathcal{P}_2^\vee(0,0,0))$ by means of the following exact sequence:

$$0 \longrightarrow p_1^*\mathcal{P}_2^\vee(-1-k,-1,-1) \longrightarrow p_1^*\mathcal{P}_2^\vee(-k,0,0) \longrightarrow \iota^* p_1^*\mathcal{P}_2^\vee(-k,0,0) \longrightarrow 0. \tag{10.5.8}$$

Once we observe that $p_1^*\mathcal{P}_2^\vee(0,0,0)$ has weight $-\omega_1+\omega_2$, by a simple application of the Borel–Weil–Bott theorem we find

$$\mathrm{Ext}^\bullet_{\mathcal{M}}(\mathcal{O}_{\mathcal{M}}(k,0,0),\iota^* p_1^*\mathcal{P}_2^\vee(0,0,0)) = 0 \text{ for } 0\le k\le 5. \tag{10.5.9}$$

Let us now mutate the second bundle of the collection 10.5.7 one step to the left. By the Euler sequence of $\mathbb{P}^5$ we find:

$$\begin{aligned}\langle \iota^*\pi^*T_{\mathbb{P}^5}(1,0,0),\mathcal{O}_{\mathcal{M}}(0,0,0),\mathcal{O}_{\mathcal{M}}(2,0,0),\dots,\mathcal{O}_{\mathcal{M}}(5,0,0),&\\ \iota^* p_1^*\mathcal{P}_2^\vee(0,0,0),\dots,\iota^* p_1^*\mathcal{P}_2^\vee(5,0,0)\rangle.&\end{aligned} \tag{10.5.10}$$

By the vanishing 10.5.8 we can move $\iota^* p_1^*\mathcal{P}_2^\vee(0,0,0)$ to the right of $\iota^*\pi^*T_{\mathbb{P}^5}(1,0,0)$. Then we consider the following short exact sequence:

$$0 \longrightarrow \iota^* p_1^*\mathcal{R}_2^\vee(0,0,0) \longrightarrow \iota^*\pi^*T_{\mathbb{P}^5}(1,0,0) \longrightarrow \iota^* p_1^*\mathcal{P}_2^\vee(0,0,0) \longrightarrow 0 \tag{10.5.11}$$

which is a consequence of the fact that $\mathcal{V} = T_{\mathbb{P}^5}(-1)$. We called $\mathcal{R}_2$ the quotient bundle of $\mathcal{G}r(2,\mathcal{V})$. By the sequence above we can mutate $\iota^* p_1^*\mathcal{P}_2^\vee(0,0,0)$ one step to the left and find:

$$\begin{aligned}\langle \iota^* p_1^*\mathcal{R}_2^\vee(0,0,0),\iota^*\pi^*T_{\mathbb{P}^5}(1,0,0),\mathcal{O}_{\mathcal{M}}(0,0,0),\mathcal{O}_{\mathcal{M}}(2,0,0),\dots,\mathcal{O}_{\mathcal{M}}(5,0,0),&\\ \iota^* p_1^*\mathcal{P}_2^\vee(1,0,0),\dots,\iota^* p_1^*\mathcal{P}_2^\vee(5,0,0)\rangle.&\end{aligned} \tag{10.5.12}$$

Then, we can mutate $\iota^*\pi^*T_{\mathbb{P}^5}(1,0,0)$ again one step to the right, finding:

$$\begin{aligned}\langle \iota^* p_1^* \mathcal{R}_2^\vee(0,0,0), \mathcal{O}_\mathcal{M}(0,0,0), \mathcal{O}_\mathcal{M}(1,0,0), \dots, \mathcal{O}_\mathcal{M}(5,0,0),\\ \iota^* p_1^* \mathcal{P}_2^\vee(1,0,0), \dots, \iota^* p_1^* \mathcal{P}_2^\vee(5,0,0)\rangle.\end{aligned} \tag{10.5.13}$$

Note that all this procedure can be applied to all the remaining twists of $\iota^* p_1^* \mathcal{P}_2^\vee$. Hence, we are left with the following collection:

$$\langle \iota^* p_1^* \mathcal{R}_2^\vee(0,0,0), \dots, \iota^* p_1^* \mathcal{R}_2^\vee(5,0,0), \mathcal{O}_\mathcal{M}(0,0,0), \dots, \mathcal{O}_\mathcal{M}(5,0,0)\rangle. \tag{10.5.14}$$

which is $\langle \iota^*[Q_2(0,0)], \iota^*[\mathcal{O}(0,0)]\rangle$. The same argument holds for mutating $\iota^*[\mathcal{U}_3^\vee(0,0)]$ through $\iota^*[\mathcal{O}(0,0)]$.

Similarly, one can compute the mutation $\mathbb{R}_{\iota^*[\mathcal{O}(-1,1)]}\iota^*[\mathcal{U}_2^\vee(0,0)]$. Again by 10.5.4 one can write the block $\langle \iota^*[\mathcal{U}_2^\vee(0,0)], \iota^*[\mathcal{O}(-1,1)]\rangle$ as:

$$\langle \iota^* p_1^* \mathcal{P}_2^\vee(0,0,0), \dots, \iota^* p_1^* \mathcal{P}_2^\vee(5,0,0), \mathcal{O}_\mathcal{M}(0,-1,1), \dots, \mathcal{O}_\mathcal{M}(5,-1,1)\rangle. \tag{10.5.15}$$

By the fact that $\iota^* p_1^* \mathcal{P}_2(-k,-1,1) \simeq \iota^* p_1^* \mathcal{P}_2^\vee(-k+1,-2,1)$ and a Borel–Weil–Bott computation, we see that $\mathrm{Ext}^\bullet_\mathcal{M}(\iota^* p_1^* \mathcal{P}_2^\vee(k,0,0), \mathcal{O}_\mathcal{M}(k,-1,1))$ vanishes for $1 \le k \le 5$, and has cohomology $\mathbb{C}[-1]$ for $k = 0$. Hence, the mutation $\mathbb{R}_{\mathcal{O}_\mathcal{M}(k,-1,1)}\iota^* p_1^* \mathcal{P}_2^\vee(0,0,0)$ can be computed by means of the (dual of the) rightmost vertical sequence of the following diagram, which can be found by applying the Snake lemma to the sequences given by the embeddings of $\mathcal{O}(-1,0,0)$ in the rank three and rank four tautological bundles of $\mathcal{Z} = F(1,3,4,6)$ (here denoted by $\mathcal{U}_3$ and

$\mathcal{U}_4$:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{O}(-1,0,0) \arrow[r] \arrow[d, equal] & p_1^*\rho^*\mathcal{U}_3 \arrow[r] \arrow[d] & p_1^*\mathcal{P}_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}(-1,0,0) \arrow[r] \arrow[d] & p_2^*\tau^*\mathcal{U}_4 \arrow[r] \arrow[d] & p_2^*\mathcal{P}_3 \arrow[r] \arrow[d] & 0 \\
 & 0 & \mathcal{O}(0,1,-1) \arrow[r, equal] \arrow[d] & \mathcal{O}(0,1,-1) \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd} \tag{10.5.16}$$

We find:

$$\langle \mathcal{O}_{\mathcal{M}}(0,-1,1), \iota^* p_2^*\mathcal{P}_3^\vee(0,0,0), \iota^* p_1^*\mathcal{P}_2^\vee(0,0,0), \dots, \iota^* p_1^*\mathcal{P}_2^\vee(5,0,0), \\ \mathcal{O}_{\mathcal{M}}(1,-1,1), \dots, \mathcal{O}_{\mathcal{M}}(5,-1,1)\rangle. \tag{10.5.17}$$

This procedure can be iterated untill all twists of $\iota^* p_1^*\mathcal{P}_2^\vee$ are mutated to twists of $\iota^* p_2^*\mathcal{P}_3^\vee$, obtaining:

$$\langle \mathcal{O}_{\mathcal{M}}(0,-1,1), \dots, \mathcal{O}_{\mathcal{M}}(5,-1,1), \iota^* p_2^*\mathcal{P}_3^\vee(0,0,0), \dots, \iota^* p_2^*\mathcal{P}_3^\vee(5,0,0), \rangle \tag{10.5.18}$$

and in this last decomposition we recognize $\langle \iota^*[\mathcal{O}(-1,1)], \iota^*[\mathcal{U}_3^\vee(0,0)]\rangle$.

11 Simple K-equivalence and roof bundles

11.1 Setup and notation

Let us recall here the notion of simple K-equivalence introduced in Chapter 3. Let X_1, X_2 be smooth projective varieties. We call K-equivalence a birational morphism

$$\mu : X_1 \dashrightarrow X_2 \tag{11.1.1}$$

such that there is the following diagram:

$$\begin{tikzcd} & X_0 \arrow[dl, "g_1"'] \arrow[dr, "g_2"] & \\ X_1 \arrow[rr, dashed, "\mu"'] & & X_2 \end{tikzcd} \tag{11.1.2}$$

where X_0 is a smooth projective variety and g_1 and g_2 are birational maps fulfilling $g_1^*K_{X_1} \simeq g_2^*K_{X_2}$. By the DK-conjecture (BO02; Kaw02), two K-equivalent varieties are expected to be derived equivalent. We can provide some evidence to this conjecture, and establish a method to verify it for the class of simple K-equivalent maps, under some assumption on the resolution X_0.

A simple K-equivalence, following the notation of Diagram 11.1.2, is a K-equivalence μ such that g_1 and g_2 are smooth blowups, which by (Li19, Lemma 2.1) have the same exceptional divisor. Then, by the structure theorem for simple K-equivalence (Kan18, Thm. 0.2), such divisor is a family of roofs of projective bundles over a smooth projective variety B. In short, we will say that μ has codimension r and exceptional divisor $\mathcal{Z}$ when $r = \operatorname{cod}_{\mathcal{X}_1} \mathcal{Z}_1 = \operatorname{cod}_{\mathcal{X}_2} \mathcal{Z}_2$ and $\mathcal{Z}$ is the exceptional divisor of g_1 and g_2 (cfr. (Kan18, Definition 1.2)). Let us now focus our attention on the following setting:

Definition 11.1.1. *We say that a simple K-equivalence μ is homogeneous of type G/P if its exceptional divisor is a homogeneous roof bundle of type G/P over a smooth projective variety B.*

For every homogeneous simple K-equivalence μ of type G/P there exists the following diagram:

$$\begin{tikzcd}
& & \mathcal{Z} \arrow[ddll, "p_1"'] \arrow[d, hook, "f"] \arrow[ddrr, "p_2"] & & \\
& & \mathcal{X}_0 \arrow[dl, "g_1"'] \arrow[dr, "g_2"] & & \\
\mathcal{Z}_1 \arrow[r, hook] \arrow[drr, "r_1"'] & \mathcal{X}_1 \arrow[rr, dashed, "\mu"] & & \mathcal{X}_2 & \mathcal{Z}_2 \arrow[l, hook'] \arrow[dll, "r_2"] \\
& & B & &
\end{tikzcd} \tag{11.1.3}$$

which is a simple adaptation of (Kan18, Diagram 0.2.1) to our setting.

11.2 Simple K-equivalence and mutations

A relation between a pair of Calabi–Yau fibrations, a homogeneous roof bundle of type G/P and a simple K-equivalence is established in the following way. Given a homogeneous simple K-equivalence of type G/P as in Diagram 11.1.3, by (Kan18, Theorem 0.2), the exceptional divisor admits two projective bundle structures $\mathcal{Z} = \mathbb{P}(\mathcal{E}_1) = \mathbb{P}(\mathcal{E}_2)$ where $\mathcal{E}_i$ is the conormal bundle of $\mathcal{Z}_i \subset \mathcal{X}_i$. By (Kan18, Proposition 1.4), one has $\mathcal{L} := \mathcal{O}(-\mathcal{Z})|_{\mathcal{Z}}$ where $p_{1*}\mathcal{L} = \mathcal{E}_1$ and $p_{2*}\mathcal{L} = \mathcal{E}_2$. Moreover, $\mathcal{Z}$ is a homogeneous roof bundle of type G/P. Assume $\mathcal{L}$ is basepoint-free and call $\mathcal{M}$ a smooth zero locus of a section Σ of $\mathcal{L}$. Then, by Lemma 4.4.5, Σ defines a pair of Calabi–Yau fibrations (X_1, X_2) over B. We will call $\mathcal{T}_i$ the preimage of X_i under the restriction

$\bar{p}_i$ of p_i to $\mathcal{M}$. All this is summarized in the following diagram:

$$
\begin{tikzcd}
 & & \mathcal{T}_1 \arrow[r, hook, "m_1"] \arrow[dddll, "\bar{p}_1"'] & \mathcal{M} \arrow[d, hook, "\iota"] & \mathcal{T}_2 \arrow[l, hook', "m_2"'] \arrow[dddrr, "\bar{p}_2"] & & \\
 & & & \mathcal{Z} \arrow[d, hook, "f"] \arrow[ddll, "p_1"'] \arrow[ddrr, "p_2"] & & & \\
 & & & \mathcal{X}_0 \arrow[dl, "g_1"'] \arrow[dr, "g_2"] & & & \\
X_1 \arrow[r, hook] & \mathcal{Z}_1 \arrow[r, hook] \arrow[drr, "r_1"'] & \mathcal{X}_1 \arrow[rr, dashed, "\mu"] & & \mathcal{X}_2 & \mathcal{Z}_2 \arrow[l, hook'] \arrow[dll, "r_2"] & X_2 \arrow[l, hook'] \\
 & & & B & & &
\end{tikzcd}
\tag{11.2.1}
$$

In the remainder of this chapter we will extensively use the language established in Notation 10.3.6. Let us perform the following computation for later convenience.

Lemma 11.2.1. *In the notation of Diagram 11.2.1 and Notation 10.3.6, one has $\omega_{\mathcal{X}_0}|_{\mathcal{Z}} \simeq \mathcal{L}^{\otimes(-r+1)} \otimes \pi^* T$ for some line bundle $T \in D^b(B)$.*

Proof. We recall that by (Ful98, Example 3.2.11) one has $\omega_{\mathcal{Z}} \simeq p_i^* \omega_{\mathcal{Z}_i} \otimes p_i^* \det \mathcal{E}_i \otimes \mathcal{L}^{\otimes(-r)}$, and since $(r_i^{-1}(b), \mathcal{E}_i|_{r_i^{-1}(b)})$ is a Mukai pair for every $b \in B$, we get $\omega_{\mathcal{Z}} \simeq \mathcal{L}^{\otimes(-r)}$ up to twists by pullbacks of line bundles from B. By the normal bundle sequence:

$$0 \longrightarrow T_{\mathcal{Z}} \longrightarrow f^* T_{\mathcal{X}_0} \longrightarrow \mathcal{N}_{\mathcal{Z}|\mathcal{X}_0} \longrightarrow 0 \tag{11.2.2}$$

one finds $f^*\omega_{\mathcal{X}_0} = \omega_{\mathcal{Z}} \otimes \det \mathcal{N}^\vee_{\mathcal{Z}|\mathcal{X}_0} = \mathcal{L}^{\otimes(-r)} \otimes \det \mathcal{N}^\vee_{\mathcal{Z}|\mathcal{X}_0}$. The proof is completed by the fact that, since $\mathcal{Z}$ is the exceptional divisor of the blowup $g_i : \mathcal{X}_0 \longrightarrow \mathcal{X}_i$, by (Kan18, Proposition 1.4) one has $\mathcal{N}_{\mathcal{Z}|\mathcal{X}_0} \simeq f^*\mathcal{O}(\mathcal{Z}) \simeq \mathcal{L}^\vee$. □

Lemma 11.2.2. *In the language of Notation 10.3.6, assume that for either $i = 1$ or $i = 2$ there is a semiorthogonal decomposition $D^b(\mathcal{X}_0) = \langle \sigma D^b(\mathcal{X}_i), f_*[W_1], \ldots, f_*[W_n]\rangle$ where every W_j is a homogeneous vector bundle on G/P and σ is a fully faithful functor. Then one has:*

$$D^b(\mathcal{X}_0) = \langle \sigma D^b(\mathcal{X}_i) \otimes \mathcal{O}(-t\mathcal{Z}), f_*[W_1 \otimes L^{\otimes t}], \ldots, f_*[W_n \otimes L^{\otimes t}]\rangle \tag{11.2.3}$$

for every $t \in \mathbb{Z}$.

Proof. Let us start from the decomposition

$$D^b(\mathcal{X}_0) = \langle \sigma D^b(\mathcal{X}_i), f_*[W_1], \ldots, f_*[W_n]\rangle. \tag{11.2.4}$$

Note that, since $\mathcal{Z}$ is the exceptional divisor of the blowup $\mathcal{X}_0 \longrightarrow \mathcal{X}_i$ and $\mathcal{L}$ is the Grothendieck line bundle of $\mathcal{Z}$ as a projective bundle over $\mathcal{Z}_i$, by (Kan18, Proposition 1.4) one has $f^*\mathcal{O}(-\mathcal{Z}) \simeq \mathcal{L}$. Hence, for any integer t, let us twist the collection above by $\mathcal{O}(-t\mathcal{Z})$, obtaining:

$$\begin{aligned} D^b(\mathcal{X}_0) = \langle &\sigma D^b(\mathcal{X}_i) \otimes \mathcal{O}(-t\mathcal{Z}), \\ &f_*[W_1] \otimes \mathcal{O}(-t\mathcal{Z}), \ldots, f_*[W_n] \otimes \mathcal{O}(-t\mathcal{Z})\rangle. \end{aligned} \tag{11.2.5}$$

We conclude the proof by showing that for every j one has $f_*[W_j] \otimes \mathcal{O}(-t\mathcal{Z}) \simeq f_*[W_j \otimes L^{\otimes t}]$. This assertion follows simply by projection formula. In fact, for every $\mathcal{G} \in D^b(B)$ we have

$$\begin{aligned} f_*(\mathcal{W}_j \otimes \pi^*\mathcal{G}) \otimes \mathcal{O}(-t\mathcal{Z}) &\simeq f_*(\mathcal{W}_j \otimes \pi^*\mathcal{G} \otimes f^*\mathcal{O}(-t\mathcal{Z})) \\ &\simeq f_*(\mathcal{W}_j \otimes \pi^*\mathcal{G} \otimes \mathcal{L}^{\otimes t}). \end{aligned} \tag{11.2.6}$$

□

Lemma 11.2.3. *In the language of Notation 10.3.6 there are the following semiorthogonal decompositions:*

$$\begin{aligned} D^b(\mathcal{X}_0) &= \langle \widetilde{g}_1^* D^b(\mathcal{X}_1), f_*[E_1], \dots, f_*[E_n] \rangle \\ &= \langle \widetilde{g}_2^* D^b(\mathcal{X}_2), f_*[F_1], \dots, f_*[F_n] \rangle. \end{aligned} \tag{11.2.7}$$

where $\widetilde{g}_i^* := g_i^*(-) \otimes \mathcal{O}(-\mathcal{Z})$.

Proof. Let us start by applying Orlov's blowup decomposition. For $-r+1 \leq k \leq -1$ one has the following fully faithful functors:

$$\begin{aligned} F_k : D^b(\mathcal{Z}_1) &\longrightarrow D^b(\mathcal{X}_0) \\ \mathcal{F} &\longmapsto f_*(p_1^*\mathcal{F} \otimes \mathcal{L}^{\otimes k}) \end{aligned} \tag{11.2.8}$$

and the semiorthogonal decomposition:

$$D^b(\mathcal{X}_0) = \langle F_{-r+1} D^b(\mathcal{Z}_1), \dots, F_{-1} D^b(\mathcal{Z}_1), g_1^* D^b(\mathcal{X}_1) \rangle. \tag{11.2.9}$$

By applying (Sam06, Theorem 3.1) to each copy of $D^b(\mathcal{Z}_1)$ with respect to the collection $D^b(\mathcal{Z}_1) = \langle J_1, \dots, J_m \rangle$ we find:

$$\begin{aligned} D^b(\mathcal{X}_0) = \langle & f_*(\mathcal{J}_1 \otimes \mathcal{L}^{\otimes(-r+1)} \otimes \pi^* D^b(B)), \dots, f_*(\mathcal{J}_m \otimes \mathcal{L}^{\otimes(-r+1)} \otimes \pi^* D^b(B)), \\ & \qquad \vdots \qquad\qquad\qquad\qquad \vdots \\ & f_*(\mathcal{J}_1 \otimes \mathcal{L}^{\otimes(-1)} \otimes \pi^* D^b(B)), \dots, f_*(\mathcal{J}_m \otimes \mathcal{L}^{\otimes(-1)} \otimes \pi^* D^b(B)), \\ & g_1^* D^b(\mathcal{X}_1) \rangle. \end{aligned} \tag{11.2.10}$$

Let us now apply the inverse Serre functor to the whole semiorthogonal complement of $g_1^* D^b(X_1)$. Then, for $1 \leq i \leq m$ and $1 \leq j \leq r-1$ the block $f_*(\mathcal{J}_i \otimes \mathcal{L}^{\otimes(-j)} \otimes \pi^* D^b(B))$ will become $f_*(\mathcal{J}_i \otimes \mathcal{L}^{\otimes(-j)} \otimes \pi^* D^b(B)) \otimes \omega_{X_0}^{\vee}$. By the projection formula we compute:

$$\begin{aligned} & f_*(\mathcal{J}_i \otimes \mathcal{L}^{\otimes(-j)} \otimes \pi^* D^b(B)) \otimes \omega_{X_0}^{\vee} \\ = {} & f_*(\mathcal{J}_i \otimes \mathcal{L}^{\otimes(-j)} \otimes \pi^* D^b(B)) \otimes f^* \omega_{X_0}^{\vee}) \\ = {} & f_*(\mathcal{J}_i \otimes \otimes \pi^* D^b(B)) \otimes \mathcal{L}^{\otimes(r-j-1)}) \end{aligned} \tag{11.2.11}$$

where the second equality follows from Lemma 11.2.1. Substituting this in the decomposition 11.2.10 we get:

$$\begin{aligned} D^b(X_0) = \langle & g_1^* D^b(X_1), \\ & f_*(\mathcal{J}_1 \otimes \pi^* D^b(B)), \ldots\ldots\ldots\ldots\ldots\ldots, f_*(\mathcal{J}_m \otimes \pi^* D^b(B)), \\ & \qquad \vdots \qquad\qquad\qquad\qquad\qquad \vdots \\ & f_*(\mathcal{J}_1 \otimes \mathcal{L}^{\otimes(r-2)} \otimes \pi^* D^b(B)), \ldots, f_*(\mathcal{J}_m \otimes \mathcal{L}^{\otimes(r-2)} \otimes \pi^* D^b(B))\rangle. \end{aligned} \tag{11.2.12}$$

Finally, we twist the whole collection by $\mathcal{O}(-\mathcal{Z})$. By Lemma 11.2.2, what we obtain is:

$$\begin{aligned} D^b(X_0) = \langle & \widetilde{g}_1^* D^b(X_1), \\ & f_*(\mathcal{J}_1 \otimes \pi^* D^b(B) \otimes \mathcal{L}), \ldots\ldots\ldots\ldots\ldots\ldots, f_*(\mathcal{J}_m \otimes \pi^* D^b(B) \otimes \mathcal{L}), \\ & \qquad \vdots \qquad\qquad\qquad\qquad\qquad \vdots \\ & f_*(\mathcal{J}_1 \otimes \mathcal{L}^{\otimes(r-1)} \otimes \pi^* D^b(B)), \ldots, f_*(\mathcal{J}_m \otimes \mathcal{L}^{\otimes(r-1)} \otimes \pi^* D^b(B))\rangle. \end{aligned} \tag{11.2.13}$$

The result follows by comparing this decomposition with Equation 10.3.28. ∎

The following lemma and its relevance in the further computations are analogous to Lemma 10.3.4 and the role it played in the previous chapter.

Lemma 11.2.4. *Let $\pi : \mathcal{Z} \longrightarrow B$ be a flat and proper morphism of smooth projective varieties, consider a closed immersion $f : \mathcal{Z} \longrightarrow \mathcal{X}$ of codimension one, where $\mathcal{X}$ is smooth and projective. Suppose there exist admissible subcategories $\mathcal{C} \subset D^b(\mathcal{X})$, $\mathcal{D} \subset D^b(\mathcal{Z})$ and vector bundles $\mathcal{E}, \mathcal{F} \in D^b(\mathcal{Z})$ relatively exceptional over B such that one has the following strong, B-linear semiorthogonal decompositions:*

$$\begin{aligned} D^b(\mathcal{Z}) &= \langle \mathcal{D}, \mathcal{E} \otimes \pi^* D^b(B), \mathcal{F} \otimes \pi^* D^b(B) \rangle \\ D^b(\mathcal{X}) &= \langle \mathcal{C}, f_*(\mathcal{E} \otimes \pi^* D^b(B)), f_*(\mathcal{F} \otimes \pi^* D^b(B)) \rangle. \end{aligned} \tag{11.2.14}$$

Then, if $\mathcal{E}$ is $\mathcal{L}$-semiorthogonal to $\mathcal{F}$, $\mathbb{L}_{\langle \mathcal{E} \otimes \pi^* D^b(B) \rangle} \mathcal{F} \otimes \pi^* D^b(B)$ commutes with f_*.

Proof. Let us recall the functors $\Psi_{\langle \mathcal{E} \otimes \pi^* D^b(B) \rangle}$ and $\Psi^!_{\langle \mathcal{E} \otimes \pi^* D^b(B) \rangle}$ defined by Equations 10.3.6 and 10.3.7. Given the notation $L_1 := f_*$, $R_1 := f^!$, $L_2 := \Psi_{\langle \mathcal{E} \otimes \pi^* D^b(B) \rangle}$, $R_2 := \Psi^!_{\langle \mathcal{E} \otimes \pi^* D^b(B) \rangle}$ we have two adjoint pairs:

$$L_1 \colon D^b(\mathcal{Z}) \rightleftarrows D^b(\mathcal{X}) : R_1, \quad L_2 \colon D^b(B) \rightleftarrows D^b(\mathcal{Z}) : R_2. \tag{11.2.15}$$

Applying Lemma 10.2.3 to the setting above yields:

$$\epsilon_{12, L_1 A} \circ L_1 L_2 R_2 (\eta_{1,A}) = L_1(\epsilon_{2,A}) \tag{11.2.16}$$

where A is any object of $D^b(\mathcal{Z})$. Thus, the following diagram commutes:

$$\begin{tikzcd} L_1L_2R_2A \arrow[r, "L_1(\epsilon_{2,A})"] \arrow[d, "L_1L_2R_2(\eta_{1,A})"'] & L_1A \arrow[d, equal] \\ L_1L_2R_2R_1L_1A \arrow[r, "\epsilon_{12,L_1A}"] & L_1A \end{tikzcd} \tag{11.2.17}$$

If we now define the functors:

$$\begin{aligned} \Xi_{\langle f_*(\mathcal{E}\otimes\pi^*D^b(B))\rangle} &:= f_*\Psi_{\langle \mathcal{E}\otimes\pi^*D^b(B)\rangle} \\ \Xi^!_{\langle f_*(\mathcal{E}\otimes\pi^*D^b(B))\rangle} &:= \Psi^!_{\langle \mathcal{E}\otimes\pi^*D^b(B)\rangle} f^! \end{aligned} \tag{11.2.18}$$

we can rewrite Diagram 11.2.17 as

$$\begin{tikzcd} f_*\Psi_{\langle \mathcal{E}\otimes\pi^*D^b(B)\rangle}\Psi^!_{\langle \mathcal{E}\otimes\pi^*D^b(B)\rangle}\mathcal{F}\otimes\pi^*\mathcal{G} \arrow[r, "f_*\epsilon_{\mathcal{Z}}"] \arrow[d, "\beta"] & f_*(\mathcal{F}\otimes\pi^*\mathcal{G}) \arrow[d, equal] \\ \Xi_{\langle f_*(\mathcal{E}\otimes\pi^*D^b(B))\rangle}\Xi^!_{\langle f_*(\mathcal{E}\otimes\pi^*D^b(B))\rangle}f_*(\mathcal{F}\otimes\pi^*\mathcal{G}) \arrow[r, "\epsilon_{\mathcal{X}}"] & f_*(\mathcal{F}\otimes\pi^*\mathcal{G}) \end{tikzcd} \tag{11.2.19}$$

where $\beta := f_*\Psi_{\langle \mathcal{E}\otimes\pi^*D^b(B)\rangle}\Psi^!_{\langle \mathcal{E}\otimes\pi^*D^b(B)\rangle}(\eta_{1,\mathcal{F}\otimes\pi^*\mathcal{G}})$, $\epsilon_{\mathcal{Z}} := \epsilon_{2,\mathcal{F}\otimes\pi^*\mathcal{G}}$, $\epsilon_{\mathcal{X}} := \epsilon_{12,f_*(\mathcal{F}\otimes\pi^*\mathcal{G})}$. Let us now prove the following:

Claim. *The map β is an isomorphism if $\mathcal{E}$ is $\mathcal{L}$-semiorthogonal to $\mathcal{F}$.*

Proving the claim is equivalent to show that under the requirement of $\mathcal{L}$-semiorthogonality the following holds:

$$\begin{aligned} &f_*\Psi_{\langle \pi^*D^b(B)\otimes\mathcal{E}\rangle}\Psi^!_{\langle \pi^*D^b(B)\otimes\mathcal{E}\rangle}f^!f_*(\mathcal{F}\otimes\pi^*\mathcal{G}) \\ =&f_*\Psi_{\langle \pi^*D^b(B)\otimes\mathcal{E}\rangle}\Psi^!_{\langle \pi^*D^b(B)\otimes\mathcal{E}\rangle}(\mathcal{F}\otimes\pi^*\mathcal{G}). \end{aligned} \tag{11.2.20}$$

Since $\mathcal{E}$ is a vector bundle, one has:

$$\begin{aligned}
&f_*\Psi_{\langle\pi^*D^b(B)\otimes\mathcal{E}\rangle}\Psi^!_{\langle\pi^*D^b(B)\otimes\mathcal{E}\rangle}f^!f_*(\mathcal{F}\otimes\pi^*\mathcal{G})\\
=&f_*(\mathcal{E}\otimes\pi^*\pi_*\mathcal{E}xt_{\mathcal{Z}}(\mathcal{E},f^!f_*(\mathcal{F}\otimes\pi^*\mathcal{G})))\\
=&f_*(\mathcal{E}\otimes\pi^*\pi_*(\mathcal{E}^\vee\otimes f^!f_*(\mathcal{F}\otimes\pi^*\mathcal{G})))\\
=&f_*(\mathcal{E}\otimes\pi^*\pi_*(\mathcal{E}^\vee\otimes f^*f_*(\mathcal{F}\otimes\pi^*\mathcal{G})\otimes\mathcal{L}^\vee[-1]))
\end{aligned}\tag{11.2.21}$$

where we used (Huy06, Corollary 3.35). By (Huy06, Corollary 11.4) and Lemma 11.2.1 one has a distinguished triangle

$$\begin{aligned}
\mathcal{F}\otimes\pi^*\mathcal{G}\otimes\mathcal{L}[1]\longrightarrow f^*f_*(\mathcal{F}\otimes\pi^*\mathcal{G})\longrightarrow\\
\mathcal{F}\otimes\pi^*\mathcal{G}\longrightarrow\mathcal{F}\otimes\pi^*\mathcal{G}\otimes\mathcal{L}[2]
\end{aligned}\tag{11.2.22}$$

Taking the tensor product by $\mathcal{E}^\vee\otimes\mathcal{L}^\vee[-1]$ one has:

$$\begin{aligned}
\mathcal{E}^\vee\otimes\mathcal{F}\otimes\pi^*\mathcal{G}\longrightarrow\mathcal{E}^\vee\otimes f^*f_*(\mathcal{F}\otimes\pi^*\mathcal{G})\otimes\mathcal{L}^\vee[-1]\longrightarrow\\
\mathcal{E}^\vee\otimes\mathcal{F}\otimes\pi^*\mathcal{G}\otimes\mathcal{L}^\vee[-1]\longrightarrow\mathcal{E}^\vee\otimes\mathcal{F}\otimes\pi^*\mathcal{G}[1]
\end{aligned}\tag{11.2.23}$$

By applying the derived π_* we find:

$$\begin{aligned}
\ldots\longrightarrow R^m\pi_*(\mathcal{E}^\vee\otimes\mathcal{F}\otimes\pi^*\mathcal{G})\longrightarrow R^{m-1}\pi_*(\mathcal{E}^\vee\otimes f^*f_*(\mathcal{F}\otimes\pi^*\mathcal{G})\otimes\mathcal{L}^\vee)\longrightarrow\\
R^{m-1}\pi_*(\mathcal{E}^\vee\otimes\mathcal{F}\otimes\pi^*\mathcal{G}\otimes\mathcal{L}^\vee)\longrightarrow R^{m+1}(\pi_*\mathcal{E}^\vee\otimes\mathcal{F}\otimes\pi^*\mathcal{G})\longrightarrow\cdots
\end{aligned}\tag{11.2.24}$$

By $\mathcal{L}$ -semiorthogonality of $\mathcal{E}$ and $\mathcal{F}$ one has $R^m\pi_*(\mathcal{E}^\vee\otimes\mathcal{F}\otimes\pi^*\mathcal{G}\otimes\mathcal{L}^\vee)=0$ for every m, hence we get an isomorphism:

$$\pi_*(\mathcal{E}^\vee\otimes f^*f_*(\mathcal{F}\otimes\pi^*\mathcal{G})\otimes\mathcal{L}^\vee[-1])\simeq\pi_*(\mathcal{E}^\vee\otimes\mathcal{F}\otimes\pi^*\mathcal{G})\tag{11.2.25}$$

If we substituting this in Equation 11.2.21 we find:

$$\begin{aligned}
&f_*\Psi_{\langle\pi^*D^b(B)\otimes\mathcal{E}\rangle}\Psi^!_{\langle\pi^*D^b(B)\otimes\mathcal{E}\rangle}f^!f_*(\mathcal{F}\otimes\pi^*\mathcal{G})\\
=&f_*(\mathcal{E}\otimes\pi^*\pi_*(\mathcal{E}^\vee\otimes\mathcal{F}\otimes\pi^*\mathcal{G}))\\
=&f_*\Psi_{\langle\pi^*D^b(B)\otimes\mathcal{E}\rangle}\Psi^!_{\langle\pi^*D^b(B)\otimes\mathcal{E}\rangle}(\mathcal{F}\otimes\pi^*\mathcal{G}).
\end{aligned}\tag{11.2.26}$$

thus proving the claim.

We are now ready to prove the isomorphism:

$$f_*\mathbb{L}_{\langle\mathcal{E}\otimes\pi^*D^b(B)\rangle}\mathcal{F}\otimes\pi^*\mathcal{G}\simeq\mathbb{L}_{\langle f_*(\mathcal{E}\otimes\pi^*D^b(B))\rangle}f_*(\mathcal{F}\otimes\pi^*\mathcal{G}). \qquad (11.2.27)$$

Let us recall the distinguished triangle in $D^b(\mathcal{Z})$:

$$\begin{aligned}\Psi_{\langle\mathcal{E}\otimes\pi^*D^b(B)\rangle}\Psi^!_{\langle\mathcal{E}\otimes\pi^*D^b(B)\rangle}\mathcal{F}\otimes\pi^*\mathcal{G}\xrightarrow{\epsilon_{\mathcal{Z}}}\\ \longrightarrow\mathcal{F}\otimes\pi^*\mathcal{G}\longrightarrow\mathbb{L}_{\langle\mathcal{E}\otimes\pi^*D^b(B)\rangle}\mathcal{F}\otimes\pi^*\mathcal{G}.\end{aligned} \qquad (11.2.28)$$

If we apply the derived pushforward f_* we find (Huy06, proof of Corollary 2.50):

$$\begin{aligned}R^0f_*\Psi_{\langle\mathcal{E}\otimes\pi^*D^b(B)\rangle}\Psi^!_{\langle\mathcal{E}\otimes\pi^*D^b(B)\rangle}\mathcal{F}\otimes\pi^*\mathcal{G}\xrightarrow{f_*\epsilon_{\mathcal{Z}}}\\ \longrightarrow R^0f_*(\mathcal{F}\otimes\pi^*\mathcal{G})\longrightarrow R^0f_*\mathbb{L}_{\langle\mathcal{E}\otimes\pi^*D^b(B)\rangle}\mathcal{F}\otimes\pi^*\mathcal{G}\\ R^1f_*\Psi_{\langle\mathcal{E}\otimes\pi^*D^b(B)\rangle}\Psi^!_{\langle\mathcal{E}\otimes\pi^*D^b(B)\rangle}\mathcal{F}\otimes\pi^*\mathcal{G}\longrightarrow\cdots\end{aligned} \qquad (11.2.29)$$

On the other hand, one has the distinguished triangle:

$$\begin{aligned}\Xi_{\langle f_*(\mathcal{E}\otimes\pi^*D^b(B))\rangle}\Xi^!_{\langle f_*(\mathcal{E}\otimes\pi^*D^b(B))\rangle}f_*(\mathcal{F}\otimes\pi^*\mathcal{G})\xrightarrow{\epsilon_{\mathcal{X}}}\\ f_*(\mathcal{F}\otimes\pi^*\mathcal{G})\longrightarrow\mathbb{L}_{\langle f_*(\mathcal{E}\otimes\pi^*D^b(B))\rangle}f_*(\mathcal{F}\otimes\pi^*\mathcal{G})\longrightarrow\cdots\end{aligned} \qquad (11.2.30)$$

and the proof is concluded by commutativity of Diagram 11.2.19 and (GM03, page 232, Corollary 4). □

Lemma 11.2.5. *In the language of Notation 10.3.6, consider a fully faithful functor σ and a set $\{W_1,\dots,W_n\}$ of exceptional, semiorthogonal objects of $D^b(G/P)$ such that for either $i=1$ or $i=2$ one has a semiorthogonal decomposition:*

$$D^b(\mathcal{X}_0)=\langle\sigma D^b(\mathcal{X}_i),f_*[W_1],\dots,f_*[W_n]\rangle. \qquad (11.2.31)$$

Then, if W_i is L-semiorthogonal to W_{i+1} one also has:

$$\begin{aligned} D^b(\mathcal{X}_0) = \langle \sigma D^b(\mathcal{X}_i), f_*[W_1], \dots, f_*[W_{i-1}], \\ f_*[\mathbb{L}_{W_i} W_{i+1}], f_*[W_{i+1}], f_*[W_{i+2}], \dots, f_*[W_n] \rangle \\ D^b(\mathcal{X}_0) = \langle \sigma D^b(\mathcal{X}_i), f_*[W_1], \dots, f_*[W_{i-1}], \\ f_*[W_{i+1}], f_*[\mathbb{R}_{W_{i+1}} W_i], f_*[W_{i+2}], \dots, f_*[W_n] \rangle. \end{aligned} \tag{11.2.32}$$

Proof. We will only prove the statement about left mutations, the other being nearly identical. Starting from the decomposition 11.2.31 and applying (Kuz10, Corollary 2.9), one finds:

$$\begin{aligned} D^b(\mathcal{X}_0) = \langle \sigma D^b(\mathcal{X}_1), f_*[W_1], \dots, f_*[W_{i-1}], \\ \mathbb{L}_{f_*[W_i]} f_*[W_{i+1}], f_*[W_i], f_*[W_{i+2}], \dots, f_*[W_n] \rangle. \end{aligned} \tag{11.2.33}$$

By Lemma 10.3.5, since W_i is L-semiorthogonal to W_{i+1} it follows that $\mathcal{W}_i$ is $\mathcal{L}$-semiorthogonal to $\mathcal{W}_{i+1}$ (the terminology is from Notation 10.3.6), hence we can apply Lemma 11.2.4 to replace $\mathbb{L}_{f_*[W_i]} f_*[W_{i+1}]$ with $f_* \mathbb{L}_{[W_i]}[W_{i+1}]$ in the decomposition 11.2.33. By Lemma 10.3.7 we have $\mathbb{L}_{[W_i]}[W_{i+1}] \simeq [\mathbb{L}_{W_i} W_{i+1}]$ and this concludes the proof. □

Proposition 11.2.6. *In the language of Notation 10.3.6, assume that for either $i = 1$ or $i = 2$ there is a semiorthogonal decomposition $D^b(\mathcal{X}_0) = \langle \sigma D^b(\mathcal{X}_i), f_*[W_1], \dots, f_*[W_n] \rangle$ where every W_j is a homogeneous vector bundle on G/P and σ is a fully faithful functor. Then one has:*

$$D^b(\mathcal{X}_0) = \langle \mathbb{R}_{f_*[W_1]} \sigma D^b(\mathcal{X}_i), f_*[W_2], \dots, f_*[W_n], f_*[W_1 \otimes L^{\otimes(r-1)}] \rangle \tag{11.2.34}$$

Proof. Let us start by mutating $\sigma D^b(\mathcal{X}_i)$ one step to the right. We obtain:

$$D^b(\mathcal{X}_0) = \langle f_*[W_1], \mathbb{R}_{f_*[W_1]}\sigma D^b(\mathcal{X}_i), f_*[W_2], \dots, f_*[W_n]\rangle. \quad (11.2.35)$$

By the inverse Serre functor we find:

$$D^b(\mathcal{X}_0) = \langle \mathbb{R}_{f_*[W_1]}\sigma D^b(\mathcal{X}_i), f_*[W_2], \dots, f_*[W_n], f_*[W_1] \otimes \omega_{\mathcal{X}_0}^\vee\rangle. \quad (11.2.36)$$

For every $\mathcal{G} \in D^b(B)$ one has:

$$\begin{aligned} f_*(\mathcal{W}_1 \otimes \pi^*\mathcal{G}) \otimes \omega_{\mathcal{X}_0}^\vee &= f_*(\mathcal{W}_1 \otimes \pi^*\mathcal{G} \otimes f^*\omega_{\mathcal{X}_0}^\vee) \\ &= f_*(\mathcal{W}_1 \otimes \pi^*\mathcal{G} \otimes \mathcal{L}^{\otimes(r-1)} \otimes \pi^*T). \end{aligned} \quad (11.2.37)$$

where T is a line bundle on B. The first isomorphism is by projection formula and the second by Lemma 11.2.1. This shows that we can substitute $f_*[W_1] \otimes \omega_{\mathcal{X}_0}^\vee$ with $f_*[W_1 \otimes L^{\otimes(r-1)}]$ in the decomposition 11.2.36, hence proving our claim. □

We are now ready to prove the main theorem of this chapter:

Theorem 11.2.7. *Let $\mu : \mathcal{X}_1 \dashrightarrow \mathcal{X}_2$ be a homogeneous simple K-equivalence of type G/P, with exceptional divisor $\mathcal{Z}$ and $\mathcal{L} := \mathcal{O}_{\mathcal{Z}}(-\mathcal{Z})$, where the pair $(\mathcal{Z}, \mathcal{L})$ satisfies Assumption 10.3.10 in the sense of Definition 10.3.11. Then μ satisfies the DK conjecture, i.e. there is an equivalence of categories $\rho : D^b(\mathcal{X}_1) \longrightarrow D^b(\mathcal{X}_2)$.*

Proof. Consider the data of Diagram 11.1.3 defining a homogeneous simpke K-equivalence μ of type G/P. Let us give a modification of the sequence of pairs $(\boldsymbol{E}^{(\lambda)}, \psi^{(\lambda)})$ introduced in Notation 10.3.6: we consider the same ordered sequence $\boldsymbol{E}^{(\lambda)} = (E_1^{(\lambda)}, \dots, E_n^{(\lambda)})$, and redefine the functor $\rho^{(\lambda)}$ to be compatible with the present setting. In analogy with the operations O1, O2, O3 of Notation 10.3.6, we define $(\boldsymbol{E}^{(\lambda+1)}, \rho^{(\lambda+1)})$ to be obtained by one of the following operations:

O4 exchange $E_i^{(\lambda)}$ with $E_{i+1}^{(\lambda)}$ and replace either $E_{i+1}^{(\lambda)}$ with $\mathbb{L}_{E_i^{(\lambda)}} E_{i+1}^{(\lambda)}$ or $E_i^{(\lambda)}$ with $\mathbb{R}_{E_{i+1}^{(\lambda)}} E_i^{(\lambda)}$, where $E_i^{(\lambda)}$ is L-semiorthogonal to $E_{i+1}^{(\lambda)}$, leave $\rho^{(\lambda)}$ unchanged

O5 move $E_1^{(\lambda)}$ right after $E_n^{(\lambda)}$ and twist it by $L^{\otimes(r-1)}$, replace $\rho^{(\lambda)}$ with $\mathbb{R}_{f_*[E_1^{(\lambda)}]}\rho^{(\lambda)}$, or, conversely, move $E_n^{(\lambda)}$ right before $E_1^{(\lambda)}$, twist it by $L^{\otimes(-r+1)}$ and replace $\rho^{(\lambda)}$ with $\mathbb{L}_{f_*[E_n^{(\lambda)}]}\rho^{(\lambda)}$

O6 for any $t \in \mathbb{Z}$, replace $E_i^{(\lambda)}$ with $E_i^{(\lambda)} \otimes L^{\otimes t}$ for all i and replace $\rho^{(\lambda)}$ with $T_{-t}\rho^{(\lambda)}$, where T_{-t} is the twist functor by $\mathcal{O}(-t\mathcal{Z})$

Moreover, we recall that $E_i^{(1)} = E_i$, $E^{(R+1)} = F_i$ for $1 \leq i \leq n$ and we impose $\rho^{(1)} = \widetilde{g}_1^*$. In light of Lemma 11.2.3, we can prove our claim by showing that for $1 \leq \lambda \leq R+1$ there is the following semiorthogonal decomposition:

$$D^b(\mathcal{X}_0) = \langle \rho^{(\lambda)} D^b(\mathcal{X}_1), f_*[E_1^{(\lambda)}], \dots, f_*[E_n^{(\lambda)}] \rangle \tag{11.2.38}$$

In fact, if this claim is true, for $\lambda = R+1$ we obtain a semiorthogonal decomposition

$$D^b(\mathcal{X}_0) = \langle \rho^{(R+1)} D^b(\mathcal{X}_1), f_*[F_1], \dots, f_*[F_n] \rangle \tag{11.2.39}$$

and, once we compare it with the semiorthogonal decompositions found by Lemma 11.2.3, we conclude setting $\rho = \rho^{(R+1)}$.

The existence of 11.2.38 for every λ can be proved by induction.

For $\lambda = 1$ the decomposition 11.2.38 exists by Lemma 11.2.3. Let us now suppose that 11.2.38 exists for $\lambda = \lambda_0$ as well, we will show the existence of such decomposition for $\lambda = \lambda_0 + 1$ by mutating the former accordingly. In fact, the pair $(\rho^{(\lambda_0+1)}, \boldsymbol{E}^{(\lambda_0+1)})$ is obtained by $(\rho^{(\lambda_0)}, \boldsymbol{E}^{(\lambda_0)})$ by one of the three operations O4, O5 and O6 and all of them induce a mutation on the decomposition 11.2.38 for $\lambda = \lambda_0$, hence they define a new semiorthogonal decomposition

$$D^b(\mathcal{X}_0) = \langle \rho^{(\lambda_0+1)} D^b(\mathcal{X}_1), f_*[E_1^{(\lambda_0+1)}], \dots, f_*[E_n^{(\lambda_0+1)}] \rangle \qquad (11.2.40)$$

as we proved by previous results. More precisely, Operation O4 gives rise to a semiorthogonal decomposition by Lemma 11.2.4, Operation O5 has been treated in Proposition 11.2.6 and Operation O6 follows by Lemma 11.2.2. Therefore the semiorthogonal decomposition 11.2.40 exists and this concludes the proof. □

The following corollary provides an extension to the results of (BO95; Kaw02; Nam03) on derived equivalence for varieties related by K-equivalence of type $A_n \times A_n$ and A_n^M which are respectively standard flops and Mukai flops.

Corollary 11.2.8. *Let $\mu : \mathcal{X}_1 \dashrightarrow \mathcal{X}_2$ be a homogeneous simple K-equivalence of type G/P, where G/P is a roof of type A_n^M, $A_n \times A_n$, A_4^G, C_2 or G_2. Suppose that $\mathcal{L}$ satisfies Assumption A1. Then $\mathcal{X}_1$ and $\mathcal{X}_2$ are derived equivalent.*

Proof. We observe that by Lemma 10.3.14, if we specialize the data of Notation 10.3.6 to roof bundles of types A_n^M, $A_n \times A_n$, A_4^G, C_2 or G_2, Assumptions A2 and A3 are satisfied. Then, the proof follows directly from Theorem 11.2.7. □

12 The middle orbit of the $GL(6)$ action on $\wedge^3\mathbb{C}^6$

12.1 A projectively self-dual singular variety

In the present chapter, we will introduce some constructions related to the fourteen-dimensional subvariety W of $\mathbb{P}^{19}$ described as the space of three-forms on a six-dimensional vector space which are partially decomposable, i.e. they can be written as the wedge product of a vector with a two-form. There are several properties of W which make it peculiar:

- W is a singular variety in codimension five, and the singular locus is the subvariety $G(3,6)$ of simple three-forms (Don77).
- There exist three desingularizations of W, of which two are crepant. All three of them are projective bundles over homogeneous varieties (BFMT17, Section 3). The non-crepant resolution can be related to a roof bundle construction, and this gives a non-trivial example of derived equivalent Calabi–Yau varieties of dimension nine as we will illustrate later in this chapter.
- W is a self-projective dual variety. This self-duality could possibly extend to homological projective duality. This last feature is extremely useful for our interests: for instance, $G(2,5)$ and $OG(5,10)^+$ are homological projective self-duals and this allows to construct derived equivalent pairs of Calabi–Yau varieties of dimension respectively three (OR17; BCP20) and five (Man17)

as intersections of general translates of such varieties.

The property of being homological projective self-dual would make W a promising candidate to extend the Calabi–Yau duality described in (OR17; BCP20; Man17) of intersections of general translates to new examples. However, a major obstruction comes from the fact that W is singular. Two approaches can be adopted to overcome the problem:

- Consider the intersection of W with the image gW under $g \in \mathrm{Aut}(\mathbb{P}^{19})$, then find a resolution of the intersection. The advantage of this approach is that the self-HPD property of W allows to use a powerful argument to prove derived equivalence of intersections. However, finding a crepant resolution of $W \cap gW$ is a challenging problem.
- Choose a desingularization of W_0 and find an embedding in an ambient variety $\mathcal{X}_0$, then consider the intersection of two translates of W_0 with respect to some automorphism group of $\mathcal{X}_0$. The main pros of this latter approach are the fact that all varieties are smooth and that a good candidate for $\mathcal{X}_0$ can be found rather easily. However, there is the nontrivial disadvantage that the desingularization W_0 is not self-homological projective dual.

In the remainder of this chapter, after giving a more precise description of W and its properties, we will construct a pair of Calabi–Yau ninefolds described as zero loci of sections of vector bundles on W_1 and W_2, which can be described as an example of pair of derived equivalent Calabi–Yau fibrations related to a homogeneous roof bundle.

12.2 Desingularizations

Let V_6 be a vector space of dimension six. There exist three desingularizations of W, appearing in the following diagram (BFMT17):

$$\begin{tikzcd}
 & W_0 \simeq \mathbb{P}(\wedge^2\mathcal{P}(-1,0)) \arrow[dl] \arrow[d, "p"] \arrow[dr] & \\
W_1 \simeq \mathbb{P}(\wedge^3 Q) \arrow[d] & F(1,5,V_6) \arrow[dl, "q"] \arrow[dr, "r"] & W_2 \simeq \mathbb{P}(\wedge^3\mathcal{U}^\vee) \arrow[d] \\
G(1,V_6) & & G(5,V_6)
\end{tikzcd} \tag{12.2.1}$$

where W_1 and W_2 are crepant resolutions and have Picard number two. However, we shall focus on W_0, of Picard number three. We call $\mathcal{U}$ the tautological bundle of $G(5,V_6)$ and Q the quotient bundle of $G(1,V_6)$. Moreover, we define $O(a,b) := q^*O(a) \otimes r^*O(b)$ for every integers a,b. $\mathcal{P}$ is defined as the cokernel of the inclusion between the pullbacks of the tautological bundles of the two Grassmannians, as in the following exact sequence on $F(1,5,V_6)$:

$$0 \longrightarrow O(-1,0) \longrightarrow r^*\mathcal{U} \longrightarrow \mathcal{P} \longrightarrow 0 \tag{12.2.2}$$

On W_0, we will use the notation $O(a,b,0) = p^*O(a,b)$, while we will call $O(0,0,1)$ the Grothendieck line bundle associated to the projective bundle structure $W_0 = \mathbb{P}(\wedge^2\mathcal{P}(-1,0))$.

Remark 12.2.1. The two crepant desingularizations give us the possibility to compute the canonical class of W by computing the canonical class of one of them, say W_1. In fact, by the relative tangent bundle sequence of the surjection $\mathbb{P}(\wedge^3 Q)) \longrightarrow G(1, V_6)$ one has $\det(T_{W_1}) = \det(RT) \otimes \mathcal{O}_{\mathbb{P}^5}(6)$, where RT is the relative tangent bundle. On the other hand, one has the relative Euler sequence associated to the projective bundle structure $W_1 = \mathbb{P}(\wedge^3 Q)$ which gives $\det RT = \mathcal{O}_{\mathbb{P}^5}(6) \otimes \mathcal{O}_{W_1}(10)$, and this allows us to conclude that W_1 (and then W) have index 10.

12.3 Tangent bundles of Grassmannians

In the next section, for later use, we are going to give an explicit description of the normal sheaf of W. Since W is singular, the outcome will not be a locally trivial fibration: however, we can get a vector bundle by pulling it back to the resolution W_0. As we will see, smooth Calabi–Yau ninefolds can be constructed as zero loci of sections of such bundle. This construction has been explained to the author by Laurent Manivel, to whom the author is greatly indebted.

Hereafter, let us remind a very explicit description of the tangent bundles of Grassmannians. Although all the material in this section is well-known, it will be useful to review it in order to fix the notation, since we will describe the tangent bundle of W as a generalization of such object. Let us fix a vector space V_n of dimension n. Explicitly, $G(k, V_n)$ can be described as the space of totally decomposable k-forms on V_n, where we say that a form $\omega \in \wedge^k V_n$ is totally decomposable if there exist k vectors $v_1, \dots, v_k$ such that $\omega = v_1 \wedge \cdots \wedge v_k$. Call $p \in G(k, V_n)$ the point associated to the vector space $Z^{(p)} = \mathrm{Span}(\{v_1, \dots, v_k\})$. Such vector space can be also associated to a totally decomposable k-form $\omega^{(p)} = v_1 \wedge \cdots \wedge v_k$ up to rescaling by a nonzero constant. Consider a linear map $H : Z^{(p)} \longrightarrow V_n/Z^{(p)}$. We can define a curve in $G(k, V_n)$ as follows:

$$\begin{aligned} \mathcal{H}_p : I &\longrightarrow G(k, V_n) \\ t &\longmapsto (v_1 + tHv_1) \wedge \cdots \wedge (v_k + tHv_k) \end{aligned} \tag{12.3.1}$$

Observe that $\mathcal{H}_p(0) = p$. We get a tangent vector to p if we consider the differentiation of the curve evaluated in $t = 0$:

$$\frac{d}{dt}\mathcal{H}_p(t)\bigg|_{t=0} = Hv_1 \wedge v_2 \wedge \cdots \wedge v_k + \cdots + v_1 \wedge \cdots \wedge v_{k-1} \wedge Hv_k \in T_pG(k, V_n). \tag{12.3.2}$$

Since we obtain a linearly independent tangent vector for each linearly independent choice of a maximal rank H, we conclude that $\operatorname{rk} T_{G(k,V_n)} = \dim \operatorname{Hom}(Z^{(p)}, V_n/Z^{(p)}) = k(n-k)$.

Remark 12.3.1. Let us fix $k = 2$: we can find an especially nice description of the normal bundle of $G(2, V_n)$, which is the one we used in Chapter 7 to discuss Calabi–Yau threefolds in $G(2, V_5)$. In fact, for every $p \in G(2, V_n)$, the orthogonal complement of $T_pG(2, V_n)$ is isomorphic to $\wedge^2(V_n/Z^{(p)})$. Taking into account the right action of $GL(V_n/Z^{(p)})$, which encodes change of basis, we get $\mathcal{N}_{G(2,V_n)|\mathbb{P}(\wedge^2 V_n)} \simeq \wedge^2 Q(1)$.

12.4 A special vector bundle on W_0

Let us come back to the variety W of partially decomposable three-forms in $\mathbb{P}(\wedge^3 V_6)$, fix a point p associated to the form $v^{(p)} \wedge \omega^{(p)}$ up to rescaling, where $v^{(p)}$ is a vector and $\omega^{(p)}$ is a two form. Extending the discussion of the previous section, we can associate curves on W to pairs (A_1, A_2) in the space

$$\mathrm{Hom}(\mathrm{Span}(v^{(p)}), V_6/\mathrm{Span}(v^{(p)})) \times \mathrm{Hom}(\mathrm{Span}(\omega^{(p)}), V_6/\mathrm{Span}(\omega^{(p)}))$$

in the following way:

$$\begin{aligned} \mathcal{A}_p : I &\longrightarrow W \\ t &\longmapsto (v^{(p)} + tA_1 v^{(p)}) \wedge (\omega^{(p)} + tA_2\omega^{(p)}) \end{aligned} \tag{12.4.1}$$

This allows us to construct the following object:

$$\hat{T}_p W := \left\{ \frac{d}{dt}\mathcal{A}_p(t)\bigg|_{t=0} \right\} \simeq V_6 \wedge \omega^{(p)} + v^{(p)} \wedge \wedge^2 V_6. \tag{12.4.2}$$

Note that this is not the fiber of a vector bundle, but rather of a piecewise locally trivial fibration over W. In fact, the dimension of $\hat{T}_p W$ jumps exactly when $\omega^{(p)}$ becomes totally decomposable, i.e. on the singular locus $\mathrm{Sing}(W) \simeq G(3, V_6)$. Let us now consider a regular point p, i.e. a point such that the associated two-form $\omega^{(p)}$ is not decomposable. Then, choosing an appropriate basis $\{e_1, \dots e_6\}$ of V_6, without loss of generality we can fix $v^{(p)} \wedge \omega^{(p)} = e_1 \wedge (e_{23} + e_{45})$ where we use the shorthand notation $e_{i_1,\dots,i_m} := e_{i_1} \wedge \dots \wedge e_{i_m}$. The associated vector space is:

$$\begin{aligned} \hat{T}_p W = \mathrm{Span}(\{&e_{245}, e_{345}, e_{234}, e_{235} + e_{456}, e_{123}, e_{124}, \\ &e_{125}, e_{126}, e_{134}, e_{135}, e_{136}, e_{145}, e_{146}, e_{156}\}). \end{aligned} \tag{12.4.3}$$

Let us now describe the orthogonal complment of $\hat{T}_p W$ in $\wedge^3 V_6$:

$$\hat{N}_p W = \mathrm{Span}(\{e_{246}, e_{256}, e_{346}, e_{356}, e_{236} - e_{456}\}). \tag{12.4.4}$$

There exists a natural embedding of vector spaces $\hat{N}_pW \subset K_p$ where:

$$K_p = \mathrm{Span}(\{e_{236}, e_{246}, e_{256}, e_{346}, e_{356}, e_{456}\}). \tag{12.4.5}$$

We can see that $K_p = \wedge^2 \mathrm{Span}(\omega^{(p)}) \wedge (\mathrm{Span}(\omega^{(p)}) \oplus \mathrm{Span}(v^{(p)}))^{\perp}$. Every regular point p defines a point in the projective bundle

$$\mathbb{P}(\wedge^2\mathcal{P}(-1,0)) \longrightarrow F(1,5,V_6).$$

In fact, the pair $\mathrm{Span}(v^{(p)}), \mathrm{Span}(\omega^{(p)}) \oplus \mathrm{Span}(v^{(p)})$ defines a point in $F(1,5,V_6)$ and the coordinate on the fiber is given by the nonzero $\omega^{(p)} \in \wedge^2\mathcal{P}(-1,0)$. This allows us to identify K_p with the fiber of $\wedge^2\mathcal{P} \otimes \mathcal{O}(1,1,1) \otimes \mathcal{O}(0,1,0) = \wedge^2\mathcal{P}(1,2,1)$. In this language, for every regular $p \in W$, one can describe $\hat{N}_pW$ as the quotient of K_p by a projection onto the orthogonal complement of a one-dimensional vector space spanned by one specific three-form. Therefore, we can describe $\hat{N}_pW$ as the fiber of a vector bundle $\mathcal{E}$ over W_0, with the following presentation:

$$0 \longrightarrow \mathcal{O}(1,1,0) \longrightarrow \wedge^2\mathcal{P}(1,2,1) \longrightarrow \mathcal{E} \longrightarrow 0 \tag{12.4.6}$$

Note that, while the vector space N_pW is defined for regular $p \in W$, $\mathcal{E}$ is a well-defined vector bundle on all W_0.

Lemma 12.4.1. *A general section of the bundle $\mathcal{E}$ defined by Equation 12.4.6 is a smooth Calabi–Yau variety of dimension nine.*

Proof. The canonical bundle of W_0 is computed in terms of the relative tangent bundle sequence with respect to p:

$$0 \longrightarrow RT \longrightarrow T_{W_0} \longrightarrow p^*T_{F(1,5,V_6)} \longrightarrow 0 \tag{12.4.7}$$

where $\det T_{F(1,5,V_6)} = \mathcal{O}(5,5)$ and the determinant of the relative tangent bundle is computed to be $\mathcal{O}(3,3,6)$ via the relative Euler sequence coming from the projective bundle p:

$$0 \longrightarrow RT^\vee(0,0,1) \longrightarrow \wedge^2\mathcal{P}(-1,0,0) \longrightarrow \mathcal{O}(0,0,1) \longrightarrow 0 \quad (12.4.8)$$

so we conclude that $\omega_{W_0} \simeq \mathcal{O}(-8,-8,-6)$. On the other hand, the determinant of $\mathcal{E}$ is given by $\mathcal{O}(6,12,6)\otimes\mathcal{O}(3,-3,0)\otimes\mathcal{O}(-1,-1,0) = \mathcal{O}(8,8,6)$. Note that $\wedge^2\mathcal{P}(1,2,1)$ is globally generated, hence $\mathcal{E}$ is globally generated as well. Finally, since W_0 has dimension fourteen and $\mathcal{E}$ has rank five, we conclude by Lemma 2.5.3. □

12.5 Some Hodge numbers computations

With the following we show that $\rho(Y) = 3$.

Lemma 12.5.1. *Let Y be as above. Then the tangent bundle of Y has the following cohomology:*

$$H^\bullet(Y, T_Y) = \mathbb{C}^{329}[-1] \oplus \mathbb{C}^3[-8]$$

Proof. The normal bundle sequence for Y is the following:

$$0 \longrightarrow T_Y \longrightarrow T_{W_0}|_Y \longrightarrow \mathcal{E}|_Y \longrightarrow 0 \quad (12.5.1)$$

where the restrictions can be computed by means of the Koszul resolution of Y in terms of wedge powers of $\mathcal{E}^\vee$. All the relevant cohomology can be obtained by Bott's theorem, since $\mathcal{E}$ is resolved by pullbacks of homogeneous bundles.

We start from the restriction of $\mathcal{E}$, which is resolved by the following sequence:

$$\begin{aligned} 0 \longrightarrow \wedge^5\mathcal{E}^\vee \otimes \mathcal{E} \longrightarrow \wedge^4\mathcal{E}^\vee \otimes \mathcal{E} \longrightarrow \wedge^3\mathcal{E}^\vee \otimes \mathcal{E} \longrightarrow \wedge^2\mathcal{E}^\vee \otimes \mathcal{E} \longrightarrow \\ \longrightarrow \mathcal{E}^\vee \otimes \mathcal{E} \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}|_Y \longrightarrow 0 \end{aligned} \tag{12.5.2}$$

Let us fix some notation: we call $\omega_1, \dots \omega_5$ the fundamental weights of the semisimple Lie algebra A_5, and $V_{\sum_i c_i\omega_i}$ will be the space of representation associated to a weight expressed in the basis of fundamental weights.

Then the cohomology of each bundle in the resolution Equation 12.5.2 can be computed by means of tensor products of wedge powers of the sequence of Equation 12.4.6. All bundles in the sequence 12.5.2 are acyclic except for the last two, which have cohomology:

$$H^\bullet(W_0, \mathcal{E}^\vee \otimes \mathcal{E}) = \mathbb{C}[0]; \quad H^\bullet(W_0, \mathcal{E}) = \left(V_{2\omega_3} \oplus V_{\omega_2+\omega_4}\right)[0]. \tag{12.5.3}$$

which gives us the result

$$H^\bullet(Y, \mathcal{E}|_Y) = \left(V_{2\omega_3} \oplus V_{\omega_2+\omega_4}\right)/\mathbb{C}[0] \simeq \mathbb{C}^{363}[0]. \tag{12.5.4}$$

Now, let us consider the tangent bundle of W_0: the cohomology of its restriction to Y can be written in terms of the Koszul resolution of Y and the sequence 12.4.7. In this last sequence the tangent bundle of $F(1,5,V_6)$ appears: its cohomology can be found using the following short exact sequences:

$$0 \longrightarrow p^*\widetilde{RT} \longrightarrow p^*T_{F(1,5,6)} \longrightarrow p^*q^*T_{G(1,V_6)} \longrightarrow 0 \tag{12.5.5}$$

$$0 \longrightarrow p^*\widetilde{RT}^\vee(1,1) \longrightarrow p^*q^*Q(1,0) \longrightarrow p^*O(1,1) \longrightarrow 0 \qquad (12.5.6)$$

where we used the fact that the flag $F(1,5,V_6)$ is isomorphic to the projectivization of $Q(1)$ over $G(1,6)$.

The cohomology of the relative tangent RT of $W_0 \longrightarrow F(1,5,V_6)$ restricted to Y can be computed by a restriction of a twist of the dual of the sequence 12.4.8, which is:

$$0 \longrightarrow O|_Y \longrightarrow \wedge^2\mathcal{P}^\vee(1,0,1)|_Y \longrightarrow RT|_Y \longrightarrow 0 \qquad (12.5.7)$$

Hence we conclude that $H^\bullet(Y, RT|_Y) \simeq \mathbb{C}[-8]$. The cohomology of $\widetilde{RT}|_Y$, in a similar fashion, can be computed restricting the sequence 12.5.6 to Y, which gives $H^\bullet(Y, \widetilde{RT}|_Y) = \mathbb{C}[0] \oplus \mathbb{C}^2[-1] \oplus \mathbb{C}[-8]$. Finally, one can compute the cohomology of the restriction of $T_{G(1,V_6)} \simeq \mathcal{U}^\vee \otimes Q$ via the Borel–Weil–Bott theorem. We get $H^\bullet(Y, p^*q^*T_{G(1,V_6)}) \simeq V_{\omega_1+\omega_5}[0] \oplus \mathbb{C}[-8]$.

All those cohomologies arrange to

$$H^\bullet(Y, T_{W_0}|_Y) = V_{\omega_1+\omega_5}/\mathbb{C}[0] \oplus \mathbb{C}^3[-8]$$

and this result, combined with 12.5.4, give the desired result, once we observe that $\dim(V_{2\omega_3} \oplus V_{\omega_2+\omega_4}/V_{\omega_1+\omega_5}) = 175 + 189 - 35 = 329$. □

Remark 12.5.2. Observe that the space of sections of $\mathcal{E}$, by Bott's theorem and Equation 12.4.6, is isomorphic to $V_{2\omega_3} \oplus V_{\omega_2+\omega_4}$. Quotienting out by the automorphisms of W_0 we get that Y moves in a space of dimension 328. On the other hand, the computation of $H^1(Y, T|_Y)$ yields a deformation space of dimension 329. This suggests that we are describing a divisor in a bigger family of varieties. This phenomenon gets particularly interesting if we observe that a similar computation for zero loci of sections of $Q^\vee(2)$ on $G(2,5)$ yields the same result: in that case, it is clear that the subfamily describes intersections of infinitesimal translates of $G(2,5)$, i.e. zero loci of sections of the normal bundle. However, in the context of subvarieties of the desingularized middle orbit W_0, there is no obvious candidate for a general family. Since $\mathcal{E}$ has been explicitly constructed as the pullback to W_0 of the "normal bundle" of $W_{regular}$, it is reasonable to expect that the general element of the complete family is somehow described as the intersection of two general translates of desingularized middle orbits in some ambient space of dimension nineteen. In the following section, we will describe the smooth Calabi–Yau ninefolds inside W_0 as zero loci of pushforwards of a hyperplane section in a suitable roof bundle. This allows us to generate pairs of derived equivalent Calabi–Yau sextuple covers of $F(1,5,V_6)$.

12.6 A family of Mukai flops

One can see that $\mathbb{P}\mathcal{E}$ admits a second projective bundle structure. In fact we observe that:

- $\pi : W_0 \longrightarrow F(1,5,V_6)$ is a smooth extremal contraction. In particular it is a $\mathbb{P}^5$-bundle.
- The vector bundle $\mathcal{E}$ on W_0 is such that for every $x \in F(1,5,V_6)$ there exists a Mukai pair $(\pi^{-1}(x), \mathcal{E}|_{\pi^{-1}(x)})$. To clarify this, let us fix $x \in F(1,5,V_6)$. If we restrict Equation 12.4.6 to $\pi^{-1}(x) \simeq \mathbb{P}^5$

we get:

$$0 \longrightarrow \mathcal{O} \longrightarrow V \otimes \mathcal{O}(1) \longrightarrow \mathcal{E}|_{\pi^{-1}(x)} \longrightarrow 0 \tag{12.6.1}$$

where V is a six dimensional vector space. This is the sequence defining the tangent bundle of $\mathbb{P}^5$. Such vector bundle is ample and globally generated, and $\det T_{\mathbb{P}^5} \otimes \omega_{\mathbb{P}^5} \simeq \mathcal{O}$, which proves our claim.

This construction can be dualized by identifying the fiber of π with $G(5, V)$, and taking the Mukai pair defined by the dual tautological bundle of $G(5, V)$. The outcome is the following, which is a specialization to our setting of 4.4.2:

$$\begin{tikzcd}
 & \mathcal{F}l(1,5,\wedge^2\mathcal{P}(1,2,1)) \arrow[dl, "p_1"] \arrow[dr, "p_2"] & \\
Y_1 \subset \mathcal{G}r(1,\wedge^2\mathcal{P}(1,2,1)) \arrow[dr, "\pi"] & & \mathcal{G}r(5,\wedge^2\mathcal{P}(1,2,1)) \supset Y_2 \arrow[dl, "\widetilde{\pi}"] \\
 & F(1,5,V_6) &
\end{tikzcd} \tag{12.6.2}$$

where we called Y_1 the zero locus of a general section of $\mathcal{E}$. By construction, p_1 is the projective bundle morphism associated to the projectivization of $\mathcal{E}$, we define $\widetilde{\mathcal{E}} := p_{2*}\mathcal{L}$ where $\mathcal{L}$ is the line bundle such that $\pi_{1*}\mathcal{L} = \mathcal{E}$. We call Y_2 the zero locus of the section of $\widetilde{\mathcal{E}}$ such that Y_1 and Y_2 are defined by pushforwards of the same section of $\mathcal{L}$.

Theorem 12.6.1. *In Diagram 12.6.2, Y_1 and Y_2 are a pair of derived* *equivalent Calabi–Yau covers of $F(1,5,V_6)$ of degree six.*

Proof. Observe that, by construction, $\mathcal{F}l(1,5,\wedge^2\mathcal{P}(1,2,1))$ is a roof bundle of type A_5^M on $F(1,5,V_6)$. Hence, derived equivalence follows from Corollary 10.3.16. The Calabi–Yau varieties Y_1 and Y_2 are fibrations over $F(1,5,V_6)$ whose general fiber is isomorphic to the zero locus of a general section of $T_{\mathbb{P}^5}$ on $\mathbb{P}^5$, which is a collection of six points. Hence, we conclude that Y_1 and Y_2 are sextuple covers of $F(1,5,V_6)$. $\square$

13 Gauged linear sigma models

13.1 GLSM and phase transitions

The idea of gauged linear sigma models originates from the physics of string theory: in the early 90s, a lot of interest has been raised by two different classes of theories which exhibited similar features:

- the nonlinear sigma model on a Calabi–Yau variety, whose manifold of supersymmetric vacua is a Calabi–Yau variety.
- the Landau–Ginzburg model of a superpotential W, whose manifold of symmetric vacua is a point.

Such apparently independent constructions have been proved by Witten (Wit93) to be related by a phenomenon called *phase transition*: they are described by a single theory whose behaviour exhibits a dramatic change while a parameter moves between two different regions. This idea, which was already common and well established in the field of solid state physics and statistical field theory, in the setting of nonlinear sigma models and Landau–Ginzburg models took the name of *gauged linear sigma model* (GLSM).

From a merely physical perspective, a GLSM is a supersymmetric field theory whose Lagrangian contains a parameter, called *Fayet–Iliopoulos parameter* τ such that in the limit in which τ can be neglected some of the interactions can be integrated out, while in the limit for $\tau >> 1$ some other interactions can be disregarded, giving rise to two different theories. For a rigorous, physical introduction to GLSM we refer to (MS) and (Wit93). In the following we will focus on a mathematical exposition of GLSM.

13.1.1 GLSM and variation of GIT

Definition 13.1.1. *We call* gauged linear sigma model $(V, G, \mathbb{C}^*_R, W)$ *the following data:*

1. *A finite dimensional vector space V*
2. *A linear reductive group G with an action on V*
3. *A R-*symmetry*, which is an action of $\mathbb{C}^*$ on V, traditionally denoted by $\mathbb{C}^*_R$*
4. *A polynomial $W : V \longrightarrow \mathbb{C}$ called* superpotential.

Moreover, we require the following conditions to hold:

1. *The G-action and the $\mathbb{C}^*_R$-action commute on V*
2. *W is G-invariant and $\mathbb{C}^*_R$-homogeneous with positive weight*

Before defining critical loci, let us first recall the notion of *semistability*, which is standard in geometric invariant theory ((King94, Section 2) for the affine case, in particular Lemma 2.2):

Definition 13.1.2. *Let (V, G) satisfy the requirements of Definition 13.1.1. Let $\rho_\tau \in \mathrm{Hom}(G, \mathbb{C}^*)$ be a character. We call G-*semistable locus *of V with respect to ρ_τ the following set:*

$$V_\tau^{ss} = \{v \in V : \{0\} \times V \ \cap \ \overline{\{(\rho_\tau(g)^{-1}, gv) | g \in G\}} = \emptyset\} \qquad (13.1.1)$$

We call unstable locus *the set* $V_\tau^u := V \setminus V_\tau^{ss}$.

Furthermore, we call GIT quotient *of* V *by* G *with respect to* ρ_τ *the following quotient:*

$$V/\!/_\tau G := V_\tau^{ss}/G. \tag{13.1.2}$$

A very useful characterization of the semistable locus is given by the Hilbert–Mumford criterion. Among several versions of this result, with different degrees of strength (see, for example, (MFK94, Chapter 2.1) for a classical source in the context of GIT for projective varieties), the following is the most suitable for our setting:

Lemma 13.1.3. *(King94, Lemma 2.4) Let* V *be a vector space endowed with an action of a linear reductive group* G*, let* ρ_τ *be a character. Then,* $v \in V_\tau^{ss}$ *if and only if there exists a one-parameter subgroup* $H : \mathbb{C}^* \longrightarrow G$ *such that* v *is unstable with respect to the* G*-action restricted to* H*.*

Lemma 13.1.3 is very useful in practice, because computing semistability with respect to a one-parameter subgroup is particularly simple: given a one-parameter subgroup $\{h_\lambda\} \subset \mathrm{Hom}(V, \mathbb{C}^*)$, one has that $v \in V$ is unstable with respect to ρ_τ if the following conditions are satisfied:

1. the expression $\rho_\tau(h_\lambda)^{-1}$ converges to zero for $\lambda \longrightarrow 0$
2. the expression $h_\lambda v$ has a limit in V for $\lambda \longrightarrow 0$

It has been shown that for GIT quotients of normal affine variety by linear reductive groups, varying the character defines a wall-and-chamber structure:

Theorem 13.1.4. *(Hal04, Theorem 3.3) Let X be a normal, affine G-variety. The GIT-equivalence classes in* $\mathrm{Hom}(G, \mathbb{C}^*)$ *corresponding to the G-action on X are the relative interiors of the cones of a rational, polyhedral fan $\Delta^G(X)$. Such regions will be denoted* chambers.

Definition 13.1.5. *Let $(V, G, \mathbb{C}^*_R, W)$ be a GLSM. We call* phase *of the GLSM a chamber of the associated polyhedral fan.*

Definition 13.1.6. *Let $(V, G, \mathbb{C}^*_R, W, I)$ be a GLSM phase, where I is the associated chamber. We call* critical locus *of the superpotential:*

$$\mathrm{Crit}(W) := Z(dW) \tag{13.1.3}$$

where dW is the gradient of W. Moreover, we call vacuum manifold *the GIT quotient:*

$$Y_I = \mathrm{Crit}(W) /\!/_\tau G. \tag{13.1.4}$$

for any $\rho_\tau \in I$.

Following the physical nomenclature, we call *geometric phase* a GLSM phase I such that Y_I is an algebraic variety of positive dimension.
Wall-crossing can drastically change the geometry of the vacuum manifold, this phenomenon is called *phase transition*. Let us discuss a simple case in the following example.

13.1.2 Example: hypersurface of degree d in $\mathbb{P}^n$

A standard example of this construction is a GLSM geometric phase yielding a hypersurface in $\mathbb{P}^n$ (Wit93) as the vacuum manifold. Namely, let us consider a GLSM given by the following data. First, we define a superpotential:

$$\begin{aligned} V = \mathbb{C}^{n+1} \oplus \mathbb{C} &\xrightarrow{\quad W \quad} \mathbb{C} \\ (x, p) &\longmapsto pf(x) \end{aligned} \tag{13.1.5}$$

where $f \in H^0(\mathbb{P}^n, \mathcal{O}(d))$ is a regular section. Let us fix $G = \mathbb{C}^*$, and the following action on V:

$$\begin{aligned} \mathbb{C}^* \times V &\longrightarrow V \\ \lambda, (x, p) &\longmapsto \lambda x, \lambda^{-d} p. \end{aligned} \tag{13.1.6}$$

One can easily verify that W is $\mathbb{C}^*$-invariant. We also fix a R-symmetry $\mathbb{C}^*_R$ acting trivially on x and with weight two on p, hence the data above defines a GLSM.

For every $\tau \in \mathbb{R}$ let us now consider a character $\rho_\tau : \mathbb{C}^* \longrightarrow \mathbb{C}$ defined as:

$$\begin{aligned} \mathbb{C}^* &\xrightarrow{\quad \rho_\tau \quad} \mathbb{C} \\ \lambda &\longmapsto \lambda^\tau. \end{aligned} \tag{13.1.7}$$

Let us consider a one-parameter subgroup $g_t : \mathbb{C}^* \longrightarrow \mathbb{C}^*$. For $\tau > 0$ one has $\rho_\tau^{-1}(g_t) \longrightarrow 0$ for $\tau \longrightarrow 0$ if and only if $g_t(\lambda)$ is a negative power of λ, hence $V^{ss}_\rho = V \setminus Z$ where Z is given by:

$$Z = \{(x, p) \in V : x = 0\}. \tag{13.1.8}$$

Therefore $Y_\rho = \{(x, p) \in V : x \neq 0, f(x) = 0, pdf(x) = 0\}$. Since f is regular, df has trivial kernel, hence we get:

$$Y_\rho \simeq Z(f). \tag{13.1.9}$$

This GIT quotient is independent from the choice of $\tau > 0$, hence $I = \{\tau \in \mathbb{R} : \tau > 0\}$ is a phase of the GLSM.

Let us now consider characters ρ_τ for $\tau < 0$. The semistable locus is now the following:

$$V_\rho^{ss} = \{(x, p) \in V : p \neq 0\} \tag{13.1.10}$$

Note that every (x, p) can be mapped to $(\lambda_p x, 1)$ by acting with λ_p such that $\lambda_p^d = p$. We are thus left with a stabilizer isomorphic to $\mathbb{Z}_d$, the group of d-th roots of unity. Therefore, we conclude that:

$$V/\!/_\rho G = \left[\mathbb{C}^{n+1}/\mathbb{Z}_d\right]. \tag{13.1.11}$$

In order to find the critical locus, we observe that f is regular on $\mathbb{P}^n$, hence the only zero of (f, df) on $\mathbb{C}^{n+1}$ is 0, which means:

$$Y_\rho = \{0\}. \tag{13.1.12}$$

The above discussion, for $n = 4$ and $d = 5$ yields the GLSM describing two phases: a quintic Calabi–Yau threefold in $\mathbb{P}^4$ and a second phase where the critical locus is a point. This construction can be easily generalized to gauged linear sigma model descriptions of Calabi–Yau complete intersections in a toric variety. In fact, toric varieties do always possess a description as GIT quotients of vector spaces by $G = (\mathbb{C}^*)^m$ for some m (MS, Chapter 7).

13.1.3 Non-abelian GLSM

A generalization of the discussion above is given by choosing G to be non-abelian. This provides new interesting scenarios, at the price of increased complexity. Namely, one can obtain examples of GLSM featuring multiple geometric phases, sometimes even non birationally equivalent. The physics of GLSM predicts that such geometric phases should be derived equivalent, and this provided candidates for derived equivalent, non birational Calabi–Yau pairs. The first of such examples, which is the so-called Pfaffian–Grassmannian pair, appeared first in (Rød98), where the varieties have been conjectured to have the same mirror: this, according to the homological mirror symmetry conjectures, would imply that they are derived equivalent. Later, derived equivalence has been proved in (BC08), and a GLSM construction featuring the Pfaffian–Grassmannian pair has been proposed in (ADS15) alongside with a new proof of derived equivalence, closer to the physical construction.

In the following, the gauged linear sigma model we are going to describe are specifically constructed such that the critical locus on one of the two phases is isomorphic to the zero locus of a regular section of a homogeneous vector bundle over a smooth homogeneous variety. Let V be a vector space endowed with the action of a reductive linear group G such that $X = (V \setminus Z)/G$ is a smooth homogeneous variety and let $\Gamma : G \longrightarrow \mathrm{End}(W)$ be a representation, where W is a vector space. Then, the action of G on $(V \setminus Z) \times W$ allows to define a G-equivariant vector bundle $\mathcal{E}$ in the following way:

$$\begin{array}{c} \mathcal{E} := (V \setminus Z) \times^G W \\ \Big\downarrow \pi \\ X = {}^{V \setminus Z}\big/_{G} \end{array} \tag{13.1.13}$$

A section of such bundle is a map of the following kind:

$$\begin{aligned} X &\xrightarrow{\ s\ } \mathcal{E} \\ [x] &\longmapsto [x, \widehat{s}(x)] \end{aligned} \tag{13.1.14}$$

where $\widehat{s}$ is a G-equivariant function compatible with the G-action on $\mathcal{E}$. Let us now consider the following function:

$$\begin{aligned} \mathcal{E}^\vee &\xrightarrow{\ \check{s}\ } \mathbb{C} \\ [x, w] &\longmapsto w \cdot \hat{s}(x) \end{aligned} \tag{13.1.15}$$

Clearly, $\check{s}$ is an invariant function. If we assume Z has codimension at least two, $\check{s}$ extends smoothly to $V \oplus W^\vee$. Therefore, adding an R-symmetry of weight two on W and trivial on V, we obtain a GLSM data if there exists a character ρ of G such that the ρ-unstable locus of $V \oplus W$ is Z. In this case, the vacuuum manifold has a particularly simple description by the following lemma, due to Okonek:

Lemma 13.1.7 (Okonek's lemma)**.** *In the setting above, let $\check{s}$ be the superpotential defined by a regular section $s \in H^0(X, \mathcal{E})$. Then the following isomorphism holds:*

$$\mathrm{Crit}(\check{s}) \cong Z(\hat{s}). \tag{13.1.16}$$

Proof. By definition, we have

$$Z(d\check{s}) = \{(u,\lambda) \ : \ [u,\lambda] \in \mathcal{E}^\vee, \hat{s}(u) = 0, \lambda \cdot d\hat{s}(u) = 0\}. \quad (13.1.17)$$

Since s is a regular section, then $\hat{s}$ is regular. Then, since its Jacobian $d\hat{s}$ has maximal rank, $\lambda \cdot d\hat{s}(u) = 0$ if and only if $\lambda = 0$. □

13.2 GLSM for the Calabi–Yau pair of type A_{2k}^G

In the following, we present an example of a GLSM with two geometric phases given by a pair of Calabi–Yau varieties associated to a roof of type A_{2k} Choosing $k = 2$, this discussion specializes to the GLSM appearing in (KR17, Section 6), and the associated Calabi–Yau pair case is the one studied in Chapter 7.

13.2.1 The universal bundle of $G(k+1, V_{2k+1})$

In order to introduce all the necessary information, let us begin by choosing the following explicit description for the twisted tautological bundle $\mathcal{U}_{k+1}(2)$ over $G(k+1, V_{2k+1})$ and its global sections. One has:

$$\begin{tikzcd} \mathcal{U}_{k+1}(2) = {}^{\mathrm{Hom}(\mathbb{C}^{k+1}, V_{2k+1})\backslash D_k \times W}\big/_{GL(k+1)} \ni (B,v) \sim (Bg^{-1}, \det g^{-2} g v) \arrow[d, shift right] \\ G(k+1, V_{2k+1}) = {}^{\mathrm{Hom}(\mathbb{C}^{k+1}, V_{2k+1})\backslash D_k}\big/_{GL(k+1)} \ni B \sim Bg^{-1}. \arrow[u, bend right, "s"'] \end{tikzcd} \quad (13.2.1)$$

where $D_k := \{B \in \mathrm{Hom}(\mathbb{C}^{k+1}, V_{2k+1}) : \mathrm{rk}\, B \leq k\}$ and $W \simeq \mathbb{C}^{k+1}$.

For every $B \in \mathrm{Hom}(\mathbb{C}^{k+1}, V_{2k+1}) \setminus \{\mathrm{rk} < k+1\}$, one can define a global section $s \in H^0(G(k+1, V_{2k+1}), \mathcal{U}_{k+1}(2))$ as a map which acts as follows on equivalence classes:

$$s : [B] \longmapsto [B, \hat{s}(B)] \tag{13.2.2}$$

where $\hat{s} : B \in \mathrm{Hom}(\mathbb{C}^{k+1}, V_{2k+1}) \setminus \{\mathrm{rk} < k+1\} \longrightarrow W$ is a $GL(k+1)$-equivariant function satisfying the condition $\hat{s}(Bg^{-1}) = \det g^{-2} g \hat{s}(B)$. Clearly, the choice of a $\hat{s}$ with the right G-equivariancy condition fixes uniquely a section on $G(k+1, V_{2k+1})$.

We can find more information on $\hat{s}$ by understanding its image under the injective morphism

$$\mathcal{U}_{k+1}(2) \xrightarrow{\ \iota\ } V_{2k+1} \otimes \mathcal{O}(2). \tag{13.2.3}$$

In fact, let us choose an equivalence class $[B] \in G(k+1, V_{2k+1})$ in the sense of the quotient description of Diagram 13.2.1. Then the restriction $\iota_{[B]}$ of ι to the fiber of $[B]$ sends a vector $v \in \mathbb{C}^{k+1}$ to its image Bv in a $k+1$-dimensional subspace of V_{k+1} spanned by the columns of $B \in [B]$. More precisely:

$$\iota : [B, v] \longmapsto [B, Bv]. \tag{13.2.4}$$

If we now call $\iota\hat{s}$ the function acting as $\iota\hat{s}(B) = B\hat{s}(B)$ we see that $\iota\hat{s}(Bg^{-1}) = \det g^{-2} \iota\hat{s}(B)$ as expected. By this last equation we see that $\iota\hat{s}(B)$ is a vector of $2k+1$ homogeneous polynomials of degree two in the minors of B of order $k+1$, hence, if we view such polynomials as functions of the variables $\{B_{ij}\}$, they have degree $2(k+1)$. Therefore $\hat{s}(B)$ is a vector of $k+1$ homogeneous polynomials in the variables $\{B_{ij}\}$ of degree $2k+1$.

13.2.2 GLSM data and variation of GIT

Let us begin this section by defining the GLSM data in the spirit of Definition 13.1.1. We choose the vector space to be $\mathrm{Hom}(\mathbb{C}^{k+1}, V_{2k+1}) \oplus W^\vee$, equipped with the $GL(k+1)$-action:

$$\begin{aligned} G \times \mathrm{Hom}(\mathbb{C}^{k+1}, V_{2k+1}) \oplus W^\vee &\longrightarrow \mathrm{Hom}(\mathbb{C}^{k+1}, V_{2k+1}) \oplus W^\vee \\ (B, \omega) &\longmapsto (Bg^{-1}, \det g^2 \omega g^{-1}) \end{aligned}$$

where $\omega \in W^\vee$ is intended as a row-vector. We choose $\mathbb{C}^*_R$ to act trivially $\mathrm{Hom}(\mathbb{C}^{k+1}, V_{2k+1})$ and with weight two on $W^\vee$. Observe that s can be smoothly extended by zero to a map

$$\begin{aligned} \mathrm{Hom}(\mathbb{C}^{k+1}, V_{2k+1}) &\xrightarrow{\quad s \quad} \mathrm{Hom}(\mathbb{C}^{k+1}, V_{2k+1}) \times W \\ B &\longmapsto (B, \hat{s}(B)). \end{aligned} \tag{13.2.5}$$

From this data we can construct the following superpotential, which is $GL(k+1)$-invariant and $\mathbb{C}^*_R$-homogeneous with weight two:

$$\begin{aligned} \mathrm{Hom}(\mathbb{C}^{k+1}, V_{2k+1}) \oplus W^\vee &\xrightarrow{\quad \check{s} \quad} \mathbb{C} \\ B, \omega &\longmapsto \omega \cdot \hat{s}(B) \end{aligned} \tag{13.2.6}$$

Remark 13.2.1. This formulation of a GLSM fits into the physical description of (HT07). In particular, the choice of a superpotential of the form given by (HT07, Equation 2.6) can be written, in physical terms, as

$$W = \int d^2\theta \, \mathrm{tr}(PB\hat{s}(B)), \tag{13.2.7}$$

where $\omega = PB$ and $P_1, \dots P_{2k+1}$ are superfields transforming as $P \mapsto \det g^2 P$ under the gauge group, which is $U(k+1)$, and the integration is on two fermionic coordinates of the superspace.

The chamber $\tau > 0$

Let ρ_τ be the character defined by $\rho_\tau(g) = \det g^{-\tau}$ and, for the moment, let τ be strictly positive. Then, in light of Lemma 13.1.3, we can characterize the unstable locus Z_+ as the set of pairs $(B, \omega) \in \mathrm{Hom}(\mathbb{C}^{k+1}, V_{2k+_1}) \oplus W^\vee$ such that there exists a one-parameter subgroup $\{g_t\} \subset GL(k+1)$ fulfilling the following conditions:

1. The expression $\det(g_t)$ converges to zero for $t \longrightarrow 0$ (so that $\rho_\tau^{-1}(g_t)$ converges to zero)
2. The expression $g_t.(B, \omega)$ has a limit in V for $t \longrightarrow 0$

The action of $GL(k+1)$ on $\mathrm{Hom}(\mathbb{C}^{k+1}, V_{2k+_1})$ preserves the rank, hence, for every matrix $B \in \mathrm{Hom}(\mathbb{C}^{k+1}, V_{2k+1})$ of corank l there exists an element $g \in GL(k+1)$ such that Bg^{-1} has the first l columns composed entirely of zeros. The general one-parameter subgroup of $GL(k+1)$ has the form:

$$g_t = \begin{pmatrix} t^{\alpha_1} & & & \\ & t^{\alpha_2} & & \\ & & \ddots & \\ & & & t^{\alpha_{k+1}} \end{pmatrix} \tag{13.2.8}$$

Let us consider the action of g_t on a pair (B, ω) where B has no nonzero element on the first l columns. The conditions 1 and 2 on such (B, ω) are satisfied if and only if $\alpha_1, \ldots, \alpha_{k+1}$ are a solution of the following set of inequalities:

$$\begin{cases} \sum_j \alpha_j > 0 & \\ \alpha_i \leq 0 & l < i \leq k+1 \\ 2\sum_j \alpha_j - \alpha_i \geq 0 & \forall i \end{cases} \tag{13.2.9}$$

This system has solutions if and only if $l \geq 1$ (such condition allows to choose α_1 sufficiently positive to satisfy the first inequality and the last block of $k+1$ inequalities). Hence we conclude that the unstable locus for the chamber $\tau > 0$ is:

$$Z_+ = \{(B, \omega) \in \mathrm{Hom}(\mathbb{C}^{k+1}, V_{2k+1}) \oplus W^\vee : \mathrm{rk}\, B < k+1\}. \tag{13.2.10}$$

Thus the GIT quotient relative to the chamber $\tau > 0$ defines the bundle $\mathcal{U}_3^\vee(-2)$ over $G(3, V_5)$ and the vacuum manifold, due to Lemma 13.1.7, is isomorphic to the Calabi–Yau threefold $Y = Z(s)$. Moreover, the superpotential being G-invariant, the map $\check{s}_+$ in Diagram 13.2.11 is well defined:

$$\begin{tikzcd} \mathcal{U}_3^\vee(-2) \arrow[r, "\check{s}_+"] \arrow[d] & \mathbb{C} & & \mathcal{U}_3(2) \arrow[d, shift right] \\ Z(s) \subset G(3, V_5) & & & G(3, V_5). \arrow[u, bend right, "s"'] \end{tikzcd} \tag{13.2.11}$$

The chamber $\tau < 0$

Here, the unstable locus is given by pairs (B, ω) such that there exists a one-parameter subgroup g_t fulfilling the following conditions:

1. $\det(g_t)^{-1}$ converges to zero for $t \longrightarrow 0$

2. $g_t.(B,\omega)$ has a limit in V for $t \longrightarrow 0$

If we impose such conditions on a one-parameter subgroup as the one of Equation 13.2.8 acting on a *general* pair (B,ω) we obtain the following set of inequalities, which have no solution:

$$\begin{cases} \sum_j \alpha_j & < 0 \\ \alpha_i & \leq 0 \quad \forall i \\ 2\sum_j \alpha_j - \alpha_i & \geq 0 \quad \forall i \end{cases} \tag{13.2.12}$$

which tells us that the general pair (B,ω) is semistable. In order to characterize the unstable locus, we observe that the $GL(k+1)$-action not only preserves the rank in $\mathrm{Hom}(\mathbb{C}^{k+1}, V_{2k+1})$, but also in $\mathrm{Hom}(\mathbb{C}^{k+1}, V_{2k+1})\oplus W^\vee \simeq \mathrm{Hom}(\mathbb{C}^{k+1}, \mathbb{C}\oplus V_{2k+1})$ where the isomorphism is given by writing a pair (B,ω) as a $(2k+2)\times(k+1)$-matrix given by adding ω on top of B as a new row. Hence, for every pair (B,ω) there exists an element $g \in GL(k+1)$ such that Bg^{-1} has the first l rows composed entirely of zeros and $\det(g)^2\omega g^{-1}$ has the first l entries equal to zero. Imposing Conditions 1 and 2 to such pair gives rise to the following subset of the inequalities 13.2.12:

$$\begin{cases} \sum_j \alpha_j & < 0 \\ \alpha_i & \leq 0 \quad l < i \leq k+1 \\ 2\sum_j \alpha_j - \alpha_i & \geq 0 \quad l < i \leq k+1 \end{cases} \tag{13.2.13}$$

This set of inequalities has solution if and only if $l \geq 0$, which translates to the condition $\{(B,\omega) \ : \ \mathrm{rk}\, B \leq k, \ker B \cap \ker \omega \neq \{0\}\}$. On the other hand, we observe that if we impose $\omega = 0$, a pair $(B,0)$ is unstable if and only if the following system has a solution:

$$\begin{cases} \sum_j \alpha_j & < 0 \\ \quad \alpha_i & \leq 0 \quad l < i \leq k+1 \end{cases} \tag{13.2.14}$$

which is always the case, regardless of l (hence the rank of B). Summing all up, we obtain the following unstable locus:

$$Z_- = \{(B, \omega) : \ker\omega \cap \ker B \neq \{0\}\} \cup \{(B, \omega) : \omega = 0\}. \tag{13.2.15}$$

Therefore, we have:

$$V_-^{ss} = \{(B, \omega) \in \mathrm{Hom}(\mathbb{C}^{k+1}, V_{2k+1}) \oplus W^\vee : \ker\omega \cap \ker B = \{0\}; \omega \neq 0\}. \tag{13.2.16}$$

We observe that, since $\ker\omega$ is k-dimensional, the condition $\ker\omega \cap \ker B = 0$ implies $\mathrm{rk}\, B \geq k$, otherwise the kernels would intersect in a non-trivial vector space.

The critical locus of our superpotential, in the phase $\tau < 0$, is described by the following equations in V_-^{ss}:

$$Z(d\check{s}) = \begin{cases} \omega \cdot d\hat{s} = 0 \\ \quad \hat{s} = 0 \end{cases} \tag{13.2.17}$$

The request of having $\omega \neq 0$ in the kernel of the transpose of $d\hat{s}$ can be rephrased saying that the Jacobian of $\hat{s}$ has a non-trivial kernel and this is not possible if B is maximal rank. This fact, combined with the condition $\mathrm{rk}\, B \geq k$, yields $\mathrm{rk}\, B = k$, which automatically satisfies $\hat{s} = 0$.

13.2.3 The critical locus for $\tau < 0$

Hereafter we will give explicit expression for the functions $\hat{s}(B)$ via the pushforward of the general expression of a hyperplane section of the flag. This determines uniquely a section of $\mathcal{U}_{k+1}(2)$ on $G(k+1,,V_{2k+1})$ and it permits to get a better description for Equation 13.2.17.

We will adopt the convention of the summation of repeated indices in order to lighten the notation. Furthermore the square brackets encasing a set of indices will mean that a tensor is made antisymmetric with respect to permutation of those indices, namely

$$T_{[i_1,\dots i_k]} = \frac{1}{k!}\sum_{\sigma\in S_k} \epsilon_\sigma T_{\sigma(i_1)\dots\sigma(i_k)}$$

where ϵ_σ is the sign of the permutation σ.

Let us call $P = G(k,V_{2k+1}) \times G(k+1,V_{2k+1})$. In the spirit of Equation 13.2.2, a section $S \in H^0(P,\mathcal{O}(1,1))$ is determined by a $GL(k)\times GL(k+1)$-equivariant map $\widehat{S}$ in the following way:

$$\begin{aligned} P &\xrightarrow{\quad S \quad} \mathbb{C} \\ ([A],[B]) &\longmapsto ([A],[B],\widehat{S}(A,B)) \end{aligned} \tag{13.2.18}$$

where $\widehat{S}$ is defined as the map

$$\mathrm{Hom}(\mathbb{C}^k,V_{2k+1}) \oplus \mathrm{Hom}(\mathbb{C}^{k+1},V_{2k+1}) \xrightarrow{\quad \widehat{S} \quad} \mathbb{C} \tag{13.2.19}$$

which act on (A,B) as

$$\widehat{S}(A,B) = S^{j_1\dots j_{k+1} i_1\dots i_k}\psi_{j_1\dots j_{k+1}}(B)\psi_{i_1\dots i_k}(A)$$

Here we introduced the notation $\psi_{i_1\dots i_k}(A)$ for the $k\times k$ minor of A obtained choosing the rows $i_1,\dots,i_k$ and $\psi_{j_1\dots j_{k+1}}(B)$ for the $k+1\times k+1$ minor of B defined by choosing the rows $j_1,\dots,j_{k+1}$. In this way, we construct the isomprphsim $H^0(P,\mathcal{O}(1,1))\simeq\wedge^k V_{2k+1}\otimes\wedge^{k+1}V_{2k+1}$, which identifies $H^0(F(k,k+1,V_{2k+1}),\mathcal{O}(1,1))$ with a subspace of $\wedge^k V_{2k+1}\otimes\wedge^{k+1}V_{2k+1}$. Note that the functions ψ are, by definition, completely antisymmetric, thus S will be antisymmetric with respect to permutations of pairs respectively in $(i_1,\dots,i_k)$ and $(j_1,\dots,j_{k+1})$.
Let us consider $S\in H^0(F(k,k+1,V_{2k+1}),\mathcal{O}(1,1))$. For every point in $F(k,k+1,V_{2k+1})$ we can choose a representative (A,B) such that A is the matrix obtained by erasing the first column of B. Thus we can use linearity of $\psi_{j_1,\dots,j_{k+1}}(B)$ with respect to the variables B_{r1} and write $\widehat{S}$ in the two following ways, up to an overall constant:

$$\widehat{S}(A,B)=S^{j_1\dots j_{k+1}i_1\dots i_k}\psi_{[j_1\dots j_k}(A)B_{j_{k+1}]1}\psi_{i_1\dots i_k}(A); \qquad (13.2.20)$$

$$\widehat{S}(A,B)=S^{j_1\dots j_{k+1}i_1\dots i_k}\psi_{j_1\dots j_{k+1}}(B)\frac{\partial}{\partial B_{p1}}\psi_{pi_1\dots i_k}(B) \qquad (13.2.21)$$

From (13.2.20) we can define a section P of $V^\vee_{2k+1}\otimes\mathcal{O}(2)$ on $G(k,V_{2k+1})$, such that its contraction with the vector $(B_{11},\dots,B_{2k+1\,1})$ yields $\widehat{S}(A,B)$:

$$\widehat{P}_r(A)=S^{j_1\dots j_{k+1}i_1\dots i_k}\psi_{[j_1\dots j_k}(A)\delta_{k]r}\psi_{i_1\dots i_k}(A) \qquad (13.2.22)$$

where δ is the Kronecker delta. In a similar way we can obtain from (13.2.21) a section Q of $V_{2k+1}\otimes\mathcal{O}(2)$, if we note that $\{\partial_{B_{11}};\dots;\partial_{B_{2k+1\,1}}\}$ define a basis of linear functionals on $\mathcal{U}_{k+1[B]}$. We get:

$$\widehat{Q}_r(B)=S^{j_1\dots j_{k+1}i_1\dots i_k}\psi_{j_1\dots j_{k+1}}(B)\psi_{ri_1\dots i_k}(B). \qquad (13.2.23)$$

Lemma 13.2.2. *Let $S \in H^0(F(k,k+1,V_{2k+1}),\mathcal{O}(1,1))$ be a general section, consider the surjections $p : F(k,k+1,2k+1) \longrightarrow G(k,V_{2k+1})$ and $q : F(k,k+1,V_{2k+1}) \longrightarrow G(k+1,V_{2k+1})$. Then the sections of Equation 13.2.20 and 13.2.21 satisfy respectively $Z(P) \simeq Z(p_*S)$ and $Z(Q) \simeq Z(q_*S)$.*

Proof. Let us begin with Q. By Equation 13.2.4 and the relative discussion, we described the image in $H^0(G(k+1,V_{2k+1}),V_{2k+1}\otimes\mathcal{O}(2))$ of (the equivariant map defining) a section s of $\mathcal{U}_{k+1}(2)$ with the following expression:

$$\iota\widehat{s}(B) = \widehat{s}_1(B)\begin{pmatrix} B_{11} \\ . \\ . \\ . \\ B_{2k+1\,1} \end{pmatrix} + \widehat{s}_2(B)\begin{pmatrix} B_{12} \\ . \\ . \\ . \\ B_{2k+1\,2} \end{pmatrix} + \cdots + \widehat{s}_{k+1}(B)\begin{pmatrix} B_{1\,k+1} \\ . \\ . \\ . \\ B_{2k+1\;k+1} \end{pmatrix}. \tag{13.2.24}$$

Comparing the Equation 13.2.24 with 13.2.21 leads us to write the following expression for $\hat{s}(B)$:

$$\hat{s}_t(B) = S^{j_1\ldots j_{k+1}i_1\ldots i_k}\psi_{j_1\ldots j_{k+1}}(B)\frac{\partial}{\partial B_r^{\,t}}\psi_{r\,i_1\ldots i_k}(B). \tag{13.2.25}$$

This is, indeed, the equivariant function defining a section of $\mathcal{U}_{k+1}(2)$ such that $Q_r(B) = B_r^{\,t}\hat{s}_t(B)$. Since B has maximal rank it has trivial kernel, hence we identify the zero locus of Q with the one of $\hat{s}$.

Let us now focus our attention on P. First, in the spirit of Equations 13.2.3 and 13.2.4, let us consider the injective morphism $\tau : \mathcal{U}_k(2) \hookrightarrow V_{2k+1} \otimes \mathcal{O}(2)$ on $G(k,V_{2k+1})$ which acts on an equivalence class as $\tau([A,v]) = [A,Av]$. This map defines an injection of global section spaces explicitly described as

$$\hat{s}(A) \longrightarrow A\hat{s}(A). \tag{13.2.26}$$

Hence, by dualizing the associated tautological sequence, we see that a section $P \in H^0(G(k, V_{2k+1}), V_{2k+1}^\vee \otimes \mathcal{O}(2))$ is image of a section of the subbundle $Q_k^\vee(2)$ if $Q(A) \in \ker(A^T)$ for every maximal rank $A \in \mathrm{Hom}(\mathbb{C}^k, V_{2k+1})$. But this is satisfied because every polynomial ${A^T}_t{}^r \widehat{P}_r$ is a linear combination of determinants of a matrix with a repeated column. Therefore we conclude that Q is the image of a section s of $Q_k^\vee(2)$ in $H^0(G(k, V_{2k+1}), V_{2k+1}^\vee \otimes \mathcal{O}(2))$. By the same reasoning as above, s is embedded in $H^0(G(k, V_{2k+1}), V_{2k+1}^\vee \otimes \mathcal{O}(2))$ by a map such that on each fiber it reduces to an injective morphism of vector spaces. Therefore, the zero locus of Q is isomorphic to the one of s. □

In the above, we wrote $\hat{s}$ as a function defined on $\mathrm{Hom}(\mathbb{C}^{k+1}, V_{2k+1}) \backslash D_k$ with values in $\mathbb{C}^{k+1}$, but we note that, as expected, it extends by zeros to a function on all $\mathrm{Hom}(\mathbb{C}^{k+1}, V_{2k+1})$. Namely, if the rank of B is smaller than $k+1$, all the $k+1 \times k+1$ minors vanish, so $\hat{s}(B) = 0$ and smoothness follows again from Hartogs' extension theorem. Then, by inspection, we see that $\hat{s}_i$ is linear in the entries of the i-th column of B and quadratic in the entries of the other two columns.

Now let us apply what we found to Equation 13.2.17. Since on V_-^{ss} one has $\mathrm{rk}\, B = k$, let us choose a basis where the first column of B vanishes. This reduces the system of $k+1 \times 2k+1$ equations $\omega \cdot d\hat{s} = 0$ to $k+1$ homogeneous polynomials of degree $2k$ in the variables B_{ij}

for $j \neq 1$. The overall factor ω_1 appearing in each of them can be discarded since the choice of having $B_{i1} = 0$ for every j, together with the condition $\ker B \cap \ker \omega = 0$ imply $\omega_1 \neq 0$. Summing all up, the critical locus for the phase $\tau < 0$ is given by

$$\mathrm{Crit}(\check{s}) = \{(B, \omega) : \ker B \cap \ker \omega = 0; \mathrm{rk}\, B = k, \partial_{B_{1i}} \hat{s}_1 = 0\}. \quad (13.2.27)$$

Moreover, since the $2k + 1$ polynomial are independent on the entries of b_1, they can be regarded as polynomials whose variables are the $k \times k$ minors of the matrix A obtained discarding the first column from B.

Finally, computing the derivatives of (13.2.25) with respect to the entries of the first column of B, we get

$$\frac{\partial}{\partial B_{p1}} \hat{s}_1(B) = S^{j_1 \dots j_{k+1} i_1 \dots i_k} \psi_{[j_1 \dots j_k(A)} \delta_{j_{k+1}]p} \psi_{i_1 \dots i_k}(A) \qquad (13.2.28)$$

which are exactly the $2k+1$ polynomials described in Equation 13.2.22, and their zero locus is isomorphic to Y_1 by Lemma 13.2.2.

So far, we got no conditions on ω except for $\omega_1 \neq 0$: the critical locus of the superpotential in the chamber $\tau < 0$ is a bundle $\mathcal{E}$ over Y_1. However, we still have a $GL(k+1)$-action on this bundle: a matrix B with zeros in the first column is fixed by a stabilizer of $GL(k + 1)$ given by matrices of the form

$$g_n^{-1} = \begin{pmatrix} a_1 & a_2 & \dots & a_{k+1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (13.2.29)$$

with $a_r \neq 0$ for every r and all the tuples $(\omega_1, \dots, \omega_{k+1})$ with nonvanishing ω_1 lie in the same orbit with respect to this stabilizer, which acts freely on them. So the action is transitive and free. Taking the quotient of $\mathcal{E}$ with respect to the $GL(k+1)$-action yields exactly $Y_1 \subset G(k, V_{2k+1})$. This proves compatibility of our GLSM construction with the description of diagram (7.1.1).

13.3 GLSM and Calabi–Yau fibrations: the roof bundle of type A_{2k}

Let us fix a roof bundle $\mathcal{Z}$ of type $G/P = F(2, 3, V_5)$. Herefter we present a GLSM describing the zero loci X_1 and X_2 as critical loci of a superpotential w related by a phase transition. We will mainly focus our attention to the Calabi–Yau pair of Section 4.5.3, but we will keep the discussion slightly more general: hereafter we fix $B = \mathbb{P}^{2k+1}$ and consequently $\mathcal{Z} = F(1, k+1, k+2, 2k+2)$.

13.3.1 Notation

The geometry for $B = \mathbb{P}^5$ has been established in Section 4.5.2. Let us briefly adapt some aspects to the case $B = \mathbb{P}^{2k+1}$. First, let us recall the bundle $\mathcal{P}$ defined by the embedding of pullbacks of tautological bundles on $F(1, k+2, V_{2k+2})$:

$$0 \longrightarrow u^*\mathcal{U}_1 \longrightarrow t^*\mathcal{U}_{k+2} \longrightarrow \mathcal{P} \longrightarrow 0 \tag{13.3.1}$$

where $u : F(1, k+2, 2k+2) \longrightarrow G(1, 2k+2)$ and $t : F(1, k+2, 2k+2) \longrightarrow G(k+2, 2k+2)$. The tautological bundle of $G(1, V_{2k+2}) \simeq \mathbb{P}^{2k+1}$ is

$\mathcal{U}_1 = \mathcal{O}(-1)$ one has $u^*\mathcal{U}_1 = \mathcal{O}(-1,0)$ and $\mathcal{U}_{k+2}$ is the tautological bundle of $G(k+2, V_{2k+2})$. It follows that $\mathcal{P}$ has rank $k+1$ and determinant $\det(\mathcal{P}) = \mathcal{O}(1,-1)$.

Let us now consider the following GIT description of $F(1, k{+}2, V_{2k+2})$:

$$F(1, k+2, V_{2k+2}) \simeq {\mathrm{Hom}(\mathbb{C}^{k+2}, V_{2k+2}) \setminus D_{k+1}} \Big/ G \tag{13.3.2}$$

where

$$G = \left\{ \begin{pmatrix} \lambda & \times \\ 0 & h \end{pmatrix} \right\} \subset GL(k+2), \quad \lambda \in \mathbb{C}^*, \quad h \in GL(k+1). \tag{13.3.3}$$

with $\times$ denoting the entries which correspond to a nilpotent subgroup, on which we have no conditions imposed.

The quotient is taken with respect to the right G-action defines as $C \sim Cg^{-1}$. Given a $k{+}1$ dimensional vector space V_{k+1}, we can describe $\mathcal{P}(1,2)$ as a G-equivariant vector bundle over $F(1, k+2, V_{2k+2})$ in the following way:

$$\begin{array}{c} \mathcal{P}(1,2) = {\mathrm{Hom}(\mathbb{C}^{k+2}, V_{2k+2}) \setminus D_{k+1} \oplus V_{k+1}} \Big/ G \\ \Big\downarrow \\ F(1, k+2, V_{2k+2}) \end{array} \tag{13.3.4}$$

where the equivalence relation on $\mathrm{Hom}(\mathbb{C}^{k+2}, V_{2k+2}) \setminus D_{k+1} \oplus V_{k+1}$ is $(C, x) \sim (Cg^{-1}, \lambda^{-3} \det h^{-2} hx)$. In fact, since $\mathcal{O}(1,0) = t^*\mathcal{U}^\vee$ and $\mathcal{O}(0,1) = u^* \det \mathcal{U}^\vee$, the weight of $\mathcal{O}(0,1)$ under its associated one-dimensional representation is $\det g^{-1} = \lambda^{-1} \det h^{-1}$.

Following the same approach we adopted for the GLSM of a Calabi–Yau pair of type A^G_{2k}, a section s of $\mathcal{P}(1,2)$ is defined by an equivariant map $\hat{s} : \mathrm{Hom}(\mathbb{C}^{k+2}, V_{2k+2}) \longrightarrow \mathbb{C}^{k+1}$ fulfilling the equivariancy condition $s([C]) = [C, \hat{s}(C)]$. Therefore it must satisfy

$$\hat{s}(Cg^{-1}) = \lambda^{-1} \det g^{-2} h\hat{s}(C). \tag{13.3.5}$$

In order to characterize $\hat{s}$, given a point $[C] \in F(1, k+2, V_{2k+2})$ with respect to the quotient description of Equation 13.3.2, let us pick a representative C, rename v the first column of C and call B the rest of the matrix. We use the notation $C = (v|B)$ for juxtaposition. Then, observe that the function $(v|B) \longrightarrow B\hat{s}((v|B))$ transforms like the fiber of $V_{2k+2} \otimes \mathcal{O}(1,2)$ under the G-action. Moreover, since its image lies in the image of B, by the maximal rank condition on $(v|B)$ it must lie in $V_{2k+2}/\mathrm{Span}(v)$, which is the fiber of t^*Q over v, where we identify v with $t(v, B) \in G(1, V_{2k+2})$. Note that, fixing v, we recover exactly the description of the section of $\mathcal{U}_{k+1}(2)$ of Equation 13.2.2 and the discussion thereafter.

13.3.2 The model

Let us call V the vector space

$$V = \mathrm{Hom}(\mathbb{C}, V_{2k+2}) \oplus \mathrm{Hom}(\mathbb{C}^{k+1}, V_{2k+2}) \oplus V^\vee_{k+1} \tag{13.3.6}$$

endowed with the following G-action:

$$\begin{aligned} G \times V &\longrightarrow V \\ g, (v, B, x) &\longmapsto (v\lambda^{-1}, Bh^{-1}, \lambda^3 \det h^2 x h^{-1}) \end{aligned} \tag{13.3.7}$$

where g decomposes as in Equation 13.3.4. Given a smooth section $s \in H^0(F, 1, k+2, V_{2k+2}), \mathcal{P}(1,2))$ we fix our superpotential as the following G-invariant function:

$$\begin{aligned} V &\xrightarrow{\quad w \quad} \mathbb{C} \\ (v, B, x) &\longmapsto x \cdot \hat{s}(v, B) \end{aligned} \tag{13.3.8}$$

where the dot is the usual contraction $V_{k+1}^{\vee} \times V_{k+1} \longrightarrow \mathbb{C}$.
We define a family of characters

$$\begin{aligned} \rho_\tau : G &\longrightarrow \mathbb{C}^* \\ g &\longmapsto \lambda^{-\tau} \det h^{-\tau} \end{aligned} \tag{13.3.9}$$

which describes a line inside the character group. We consider the variation of GIT related to the crossing of the vertex between the loci $\tau > 0$ and $\tau < 0$. More precisely, fixed one of the two chambers, we investigate the locus $Z_\pm \in V$ of triples (v, B, x) such that there exists a one-parameter subgroup $\{g_t\} \subset G$ with $\rho_\pm^{-1}(g_t) \longrightarrow 0$ and $g_t(v, B, x)$ has a limit in V for $t \longrightarrow 0$. Then, the corresponding semistable locus is $V_\pm^{ss} = V \setminus Z_\pm$.
Let us fix a one-parameter subgroup in G depending on $k+2$ parameters $\alpha_0, \ldots, \alpha_{k+1}$ whose elements are

$$g_t = \begin{pmatrix} t^{\alpha_0} & & \\ & \ddots & \\ & & t^{\alpha_{k+1}} \end{pmatrix} \tag{13.3.10}$$

The chamber $\tau > 0$

Here the condition $\rho_+^{-1}(g_t) \longrightarrow 0$ translates to $\sum_{i=0}^{k+1} \alpha_i > 0$. Then $(v, B, x) \in Z_+$ if and only if there exist a tuple $\alpha_0, \ldots \alpha_{k+1}$ satisfying a set of inequalities which for the general (v, B, x) are:

$$\begin{cases} \sum_{i=0}^{k+1} \alpha_i & > 0 \\ -\alpha_j & \geq 0 \\ 3\alpha_0 + 2\sum_{i=1}^{k+1} \alpha_i - \alpha_j & \geq 0 \end{cases} \tag{13.3.11}$$

Then, the semistable locus can be determined following the same kind of analysis we performed in Section 13.2. We find:

$$V_+^{ss} = \{(v, B, x) \in V | \operatorname{rk} v = 1, \operatorname{rk} B = k+1\}. \tag{13.3.12}$$

Therefore, since

$$V/\!/_+G = V_+^{ss}/G = \mathcal{P}^\vee(-1, -2). \tag{13.3.13}$$

we conclude that $(w)/\!/_+G \simeq X$ by Lemma 13.1.7.

The chamber $\tau < 0$

Here the condition $\rho_-^{-1}(g_n) \longrightarrow 0$ gives the inequality to $\sum_{i=0}^{k+1} \alpha_i < 0$. The other inequalities are unchanged, but the solution is radically different:

$$V_-^{ss} = \{(v, B, x) \in V | \operatorname{rk} v = 1, \operatorname{rk} x = 1, \ker B \cap \ker x = \{0\}\}. \tag{13.3.14}$$

Acting with G we can reduce to the situation where $x = (1, 0, \ldots, 0)$. Then the stabilizer has the form

$$G_S = \left\{ g \in G : g = \begin{pmatrix} \lambda & z_{k+1} & z_{k+2} & \ldots & z_{2k+1} \\ 0 & \delta & 0 & \ldots & 0 \\ 0 & z_1 & m_{11} & & m_{1k} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & z_k & m_{k1} & \ldots & m_{kk} \end{pmatrix} \right\} \tag{13.3.15}$$

We observe that the action of the stabilizer on B preserves linear combinations of the last k columns, while the first one transforms like the image of the fiber of $t^*Q(-1,-2)$. Hence, the GIT quotient is

$$V/\!/_{-}G = V_+^{ss}/G = r^*Q^\vee(-1,-2). \tag{13.3.16}$$

13.3.3 The phase transition

In order to prove that the critical locus in the second phase is isomorphic to X_2, we need to describe the section s more explicitly. First let us describe $S \in H^0(F(1,k+1,k+2,V_{2k+2}),\mathcal{O}(1,1,1))$. In analogy with Equation 13.3.2, the flag variety $F(1,k+1,k+2,V_{2k+2})$ is given by the following GIT description:

$$F(1,k+1,k+2,V_{2k+2}) \simeq \frac{\mathrm{Hom}(\mathbb{C}^{k+2},V_{2k+2}) \setminus Z}{H} \tag{13.3.17}$$

where

$$H = \left\{ \begin{pmatrix} \lambda & \times & \times \\ 0 & h & \times \\ 0 & 0 & \delta \end{pmatrix} \right\} \subset GL(k{+}2), \quad \lambda,\delta \in \mathbb{C}^*, \quad h \in GL(k). \tag{13.3.18}$$

and the action is $C \simeq Cg^{-1}$ for every $g \in H$. Let us write $C = (v|A|u) \in \mathrm{Hom}(\mathbb{C}^{k+2},V_{2k+2})$ where $v,u \in \mathrm{Hom}(\mathbb{C},V_{2k+2})$ and $A \in \mathrm{Hom}(\mathbb{C}^2,V_{2k+2})$. Then, a section S of $\mathcal{O}(1,1,1)$ acts in the following way:

$$S((v|A|u)) = S^{ij_1\dots j_{k+1}l_1\dots l_{k+2}}\, v_i\psi_{j_1\dots j_{k+1}}(v|A)\psi_{l_1\dots l_{k+2}}(v|A|u) \tag{13.3.19}$$

where $\psi_{k_1,\dots k_r}$ is the minor obtained choosing the lines $k_1,\dots k_r$, hence the coordinates $\psi_{k_1,\dots k_r}$ defines a Plücker map to $\wedge^r V_{2k+2}$. As we did in the previous section, to unclutter the notation, we use Einstein's summation convention which omits sums over repeated high and low indices. We observe that

$$S(g.(v|A|u)) = \lambda^{-3}\det h^{-2}\delta^{-1}S((v|A|u)) \tag{13.3.20}$$

which is the correct equivariancy condition since $\mathcal{O}(1,1,1) \simeq \mathcal{O}(1) \boxtimes \mathcal{O}(1) \boxtimes \mathcal{O}(1)$. Then, the pushforwards of this section to $F(1, k+1, V_{2k+2})$ and $F(1, k+2, V_{2k+2})$ are described by the following equivariant functions:

$$\hat{\sigma}^r((v|A)) = S^{ij_1\dots j_{k+1}l_1\dots l_{k+2}}\, v_i\psi_{j_1\dots j_{k+1}}(v|A)\delta^r{}_{[j_{k+2}}\psi_{j_1\dots j_{k+1}]}(v|A) \tag{13.3.21}$$

$$\hat{s}^r((v|B)) = S^{ij_1\dots j_{k+1}l_1\dots l_{k+2}}\, v_i\left(\frac{\partial}{\partial B_r{}^t}\psi_{j_1\dots j_{k+1}t}(v|B)\right)\psi_{l_1\dots l_{k+2}}(v|B) \tag{13.3.22}$$

where square brackets around a set of indices means totally skew-symmetric. What is left to prove is that the quotient of the critical locus of w restricted to V^{ss}_{-} by G is isomorphic to X_1. Let us write the superpotential explicitly: by Equations 13.3.8 and 13.3.22 we have

$$(v, B, x) \longrightarrow x^r S^{ij_1\dots j_{k+1}l_1\dots l_{k+2}}\, v_i\frac{\partial}{\partial B_r{}^t}\left[\psi_{j_1\dots j_{k+1}t}(v|B)\right]\psi_{l_1\dots l_{k+2}}(v|B) \tag{13.3.23}$$

As we showed before, for every G_S-orbit in V^{ss}_{-} there exist a unique point such that $x = x_0 := (1, 0, \dots, 0)$. Let us work on such points. Define:

$$\widetilde{V} = \{(v, B) : \operatorname{rk} v = 1, B_r{}^1 = 0\ \forall r \leq 2k+2\}. \tag{13.3.24}$$

We are interested in the locus

$$dw \cap \widetilde{V} = \{(v, B, x) : x = x_0, (v, B) \in \widetilde{V}, \hat{s}(v, B, x) = 0, x\cdot ds(v, B, x) = 0\}. \tag{13.3.25}$$

If $(v, B) \in \widetilde{V}$ the first equation is automatically satisfied, since $\psi(v|B)$ is identically zero for lower rank matrices, and the first column of B is zero. Let us now focus on the second equation defining the critical locus. By Equation 13.3.23, restricted to $(\widetilde{V}, x_0)$ it becomes (up to sign):

$$\begin{aligned} x \cdot ds(v, B, x_0)^z|_{(v,B)\in\widetilde{V}} \\ = S^{ij_1\dots j_{k+1}l_1\dots l_{k+2}} v_i \frac{\partial}{\partial B_1^{\ t}} \left[\psi_{j_1\dots j_{k+1}t}(v|B)\right] \frac{\partial}{\partial B_1^{\ z}} \left[\psi_{l_1\dots l_{k+2}}(v|B)\right]\Big|_{(v,B)\in\widetilde{V}} \\ = S^{ij_1\dots j_{k+1}l_1\dots l_{k+2}} v_i \psi_{j_1\dots j_{k+1}}(v|\widetilde{A}) \delta^z{}_{[l_{k+2}} \psi_{l_1\dots l_{k+1}]}(v|\widetilde{A}) := R^z(A). \end{aligned} \tag{13.3.26}$$

where $\widetilde{A} \in \mathrm{Hom}(\mathbb{C}^k, V_{2k+2})$ is the matrix resulting by removing the first (vanishing) column from B. This last equation coincides with 13.3.21, hence it describes the image in $H^0(F(1, k+1, V_{2k+2}), V_{2k+2} \otimes O(1, 2))$ of a section of $r^*Q^\vee(1, 2)$ on $F(1, k+1, V_{2k+2})$. Summing all up, the critical locus of w on V_-^{ss} is a bundle over the zero locus of the $2k+2$ equations $x \cdot dw$. Exactly as in the previous section, we observe that the $2k+2$ equations vanish exactly where the associated section of $r^*Q^\vee(1, 2)$ vanish, hence the critical locus is a bundle over the Calabi–Yau variety X_1. More precisely, we apply the same reasoning as in Lemma 13.2.2: the contraction of the vector $R(A)$ with any column of A is identically zero, hence $R(A) \in \ker(A^T)$, which allows us to conclude that R is the image of a section s of $r^*Q^\vee(1, 2)$ by an injective morphism of vector bundles, hence $Z(R) \simeq Z(s)$.

The last step is to observe that the action of the stabilizer G_S described by Equation 13.3.7 is transitive and free on $\{x = (x_1, \dots, x_{k+1})\}$.

Hence, taking the quotient by G_S, we obtain the Calabi–Yau variety X_1.

If we choose $k = 2$ we obtain a GLSM description of a pair of Calabi–Yau fibrations associated to the roof bundle of type A_4^G. Hence, we can state the following theorem summarizing the dualities appearing in this picture.

Theorem 13.3.1. *The general pair (X_1, X_2) of Calabi–Yau fibrations associated to the roof bundle of type A_4^G over $\mathbb{P}^5$ is a pair of derived equivalent Calabi–Yau eightfolds of Picard number two, and given the maps $f_1 : X_1 \longrightarrow \mathbb{P}^5$ and $f_2 : X_1 \longrightarrow \mathbb{P}^5$, for general $b \in \mathbb{P}^5$ the pair $(Y_1 := f_1^{-1}(b), Y_2 := f_1^{-1}(b))$ is a pair of non birational, derived equivalent Calabi–Yau threefolds associated to the roof of type A_4^G. Moreover, X_1 and X_2 are isomorphic to the critical loci of two phases of a non abelian gauged linear sigma model.*

Proof. Let us consider the roof bundle of type A_4^G over $\mathbb{P}^5$. By the discussion of Section 4.5.3, X_1 and X_2 are Calabi–Yau eightfolds. In particular, by Lemma 4.5.1, they have Picard number two. Derived equivalence follows from Corollary 10.3.16. By the above, X_1 and X_2 are isomorphic to the critical loci of w in the two stability chambers $\tau < 0$ and $\tau > 0$. Finally, the fibers $Y_1 := f_1^{-1}(b)$ and $Y_2 := f_1^{-1}(b)$ are a Calabi–Yau pair of type A_4^M, hence, for a general M, they are non birational by Theorem 7.2.6, and derived equivalent by Proposition 9.5.7. $\square$

14 Matrix factorization categories and Knörrer periodicity

In this chapter we review the concept of matrix factorization category and how this object relates to the derived category of coherent sheaves of some varieties via the so-called Knörrer periodicity. The content of this chapter consists in part of the material covered in a series of talks given by the author at the Université Paul Sabatier in the spring 2019.

14.1 dg-categories

Definition 14.1.1. *(dg-category, (Toen11, Section 2.3)) A dg category C over k is given by the following data:*

- *A set of objects*
- *For every pair x, y of objects in C, a complex $T(x, y)$*
- *For every triple of objects x, y, z a composition morphism:*

$$\mu_{xyz} : T(x, y) \otimes T(y, z) \longrightarrow T(x, z) \tag{14.1.1}$$

- *For every object x a morphism*

$$e_x : k \longrightarrow T(x, x) \tag{14.1.2}$$

Moreover, we require the fulfillment of the following conditions:

- *Associativity: for every quadruple x, y, z, t the following diagram commutes:*

$$\begin{tikzcd}
T(x,y)\otimes T(y,z)\otimes T(z,t) \arrow[r, "\mathrm{Id}\otimes\mu_{yzt}"] \arrow[d, "\mu_{xyz}\otimes\mathrm{Id}"'] & T(x,y)\otimes T(y,t) \arrow[d, "\mu_{xyt}"] \\
T(x,z)\otimes T(z,t) \arrow[r, "\mu_{xzt}"] & T(x,t)
\end{tikzcd} \tag{14.1.3}$$

- *For every pair x, y the following compositions are equal to the identity:*

$$\begin{gathered}
T(x,y)\simeq k\otimes T(x,y) \xrightarrow{e_x\otimes\mathrm{Id}} T(x,x)\otimes T(x,y) \xrightarrow{\mu_{xxy}} T(x,y) \\
T(x,y)\simeq T(x,y)\otimes k \xrightarrow{\mathrm{Id}\otimes e_y} T(x,y)\otimes T(y,y) \xrightarrow{\mu_{xyy}} T(x,y)
\end{gathered} \tag{14.1.4}$$

Example 14.1.2. A typical example of this construction is given by the dg category of complexes, with morphism given by total complexes (see for example (Toen11, Section 2.3, Example 2)). More precisely, given two complexes of k-modules $x^\bullet$, $y^\bullet$, we can define

$$T(x^\bullet, y^\bullet) = \mathrm{Hom}^\bullet(x^\bullet, y^\bullet) \tag{14.1.5}$$

where the Hom total complex is defined in degree k as

$$\mathrm{Hom}^k(x^\bullet, y^\bullet) := \prod_{p\in\mathbb{Z}} \mathrm{Hom}(x^p, y^{p+k}) \tag{14.1.6}$$

with the boundary map given by

$$\begin{aligned}
\mathcal{D} : \mathrm{Hom}^k(x^\bullet, y^\bullet) &\longrightarrow \mathrm{Hom}^{k+1}(x^\bullet, y^\bullet) \\
\left\{f^k : x^\bullet \longrightarrow y^{\bullet+k}\right\} &\longmapsto \left\{d^{(y)}\circ f^k - (-1)^k f^{k+1}\circ d^{(x)}\right\}
\end{aligned} \tag{14.1.7}$$

Then, if we define the compositions μ simply as chain map compositions and for every object $x^\bullet$ we call e_x the (degreewise) k-module identity map, one can verify that these data fulfill the requirements of Definition 14.1.1.

We also recall the standard notion of the homotopy category of a given dsg-category:

Definition 14.1.3. *Let C be a dg-category. We call* homotopy category *of C the category $[C]$ defined by the following data:*

- *Objects: the objects of C*
- *Morphisms:* $\mathrm{Hom}_{[C]}(x^\bullet, y^\bullet) := H^0(T(x, y))$
- *Compositions of morphisms:*

$$\begin{array}{ccc} H^0(T(x,y)) \otimes H^0(T(y,z)) & \xrightarrow{\phi} & H^0(T(x,y) \otimes T(y,z)) \\ & & \Big\downarrow{\scriptstyle \mu_{xyz}} \\ & & H^0(T(x,z)) \end{array} \tag{14.1.8}$$

Remark 14.1.4. The name "homotopy category" is justified by the following standard example: let C be the dg-category of k-modules defined in Example 14.1.2. Then, for every pair $x^\bullet, y^\bullet$ of objects of C, $\mathrm{Hom}_{[C]}(x^\bullet, y^\bullet)$ describes precisely chain maps up to homotopies.

14.2 D-brane categories

While the theory of curved dg-sheaves (and the related categories) adapts perfectly to the setting of a gauged linear sigma model, it is sufficient to estblish a more general framework, which is known in literature as *Landau–Ginzburg model*. Such term is used, both in physics and mathematics, to denote several kinds of constructions related to the pair (X, W) of a variety and a superpotential. We follow the definition introduced in (Shi12, Definition 2.1):

Definition 14.2.1 (Landau–Ginzburg model)**.** *Let* $X = [\Sigma/H]$ *be a stack, where* Σ *is a variety and* H *is an Abelian group. Let* Γ *be a* $\mathbb{C}^*$*-action over* X *called* R*-symmetry. We call* Landau–Ginzburg model *a pair* (X, W) *where* $W : X \longrightarrow \mathbb{C}$ *is an equivariant function of weight two with respect to the* R*-symmetry.*

Remark 14.2.2. Despite the generality of the definition above, throughout this chapter we will only deal with examples where X is the total space of a vector bundle over a smooth projective variety and H is trivial.

Definition 14.2.3 (curved dg-sheaf)**.** *Let* (X, W) *be a Landau–Ginzburg model. We call* curved dg-sheaf *a pair* $(\mathcal{E}, d_{\mathcal{E}})$ *such that:*

- $\mathcal{E}$ *is a sheaf of* $\mathcal{O}_X$*-modules*
- $d_{\mathcal{E}}$ *is an endomorphism of* $\mathcal{E}$ *of* $\mathbb{C}^*_R$*-weight 1, such that* $d_{\mathcal{E}} \circ d_{\mathcal{E}}$ *is the multiplication by* W*.*

Definition 14.2.4 (curved dg-sheaf, alternative definition)**.** *Let* (X, W) *be a Landau–Ginzburg model. We call* curved dg-sheaf *a pair* $(\mathcal{E}, d_{\mathcal{E}})$ *such that:*

- *A pair* $\mathcal{E}_0, \mathcal{E}_1$ *of* C^*_R*-equivariant sheaves of* $\mathcal{O}_X$*-modules*

- *A pair of morphisms $P_{\mathcal{E}} : \mathcal{E}_0 \longrightarrow \mathcal{E}_1$ and $Q_{\mathcal{E}} : \mathcal{E}_1 \longrightarrow \mathcal{E}_0$ $\mathbb{C}^*_R$-weight 1 such that $P_{\mathcal{E}} \circ Q_{\mathcal{E}} = \mathrm{Id}_{\mathcal{E}_1} W$, $Q_{\mathcal{E}} \circ P_{\mathcal{E}} = \mathrm{Id}_{\mathcal{E}_0} W$*

Definition 14.2.3 and Definition 14.2.4 are equivalent. To see this, let us begin by considering the data of the former definition: since $\mathcal{E}$ is $\mathbb{C}^*_R$-equivariant, $-1 \in \mathbb{C}^*_R$ splits $\mathcal{E}$ in two "eigensheaves" described by the following rules:

$$\begin{aligned} \mathcal{E}_0 &= \{U \mapsto \mathcal{E}(U) : \mathcal{E}((-1) \cdot U) = \mathcal{E}(U)\} \\ \mathcal{E}_1 &= \{U \mapsto \mathcal{E}(U) : \mathcal{E}((-1) \cdot U) = -\mathcal{E}(U)\} \end{aligned} \tag{14.2.1}$$

Therefore, we recover the data of Definition 14.2.4 if we set $P_{\mathcal{E}} := d_{\mathcal{E}}|_{\mathcal{E}_0}$ and $Q_{\mathcal{E}} := d_{\mathcal{E}}|_{\mathcal{E}_1}$.

Remark 14.2.5. Let the $\mathbb{C}^*_R$-action be trivial on X and fix $W = 0$. Then $(\mathcal{E}, d_{\mathcal{E}})$ breaks down to a $\mathbb{Z}$-graded complex. In fact, since $d_{\mathcal{E}}$ has $\mathbb{C}^*_R$-weight 1 and $d_{\mathcal{E}} \circ d_{\mathcal{E}} = 0$, the $\mathbb{Z}$-grading is given by the $\mathbb{C}^*_R$-weight.

Definition 14.2.6. *We call* matrix factorization *a curved dg-sheaf $(\mathcal{E}, d_{\mathcal{E}})$ such that $\mathcal{E}$ is a vector bundle.*

Definition 14.2.7. *Let $X_k : \mathbb{C}^*_R \longrightarrow \mathbb{C}^*_R$ be the character such that $X_k(\lambda) = \lambda^k$ for every $\lambda \in \mathbb{C}^*_R$ and $k \in \mathbb{Z}$. Let $\mathcal{L}_k$ be the $\mathbb{C}^*_R$-equivariant line bundle associated to X_k. Then, we define* shift *by k the following operation, for every curved dg-sheaf $(\mathcal{E}, d_{\mathcal{E}})$:*

$$(\mathcal{E}, d_{\mathcal{E}}) \longmapsto (\mathcal{E}[k], d_{\mathcal{E}[k]}) := (\mathcal{E} \otimes \mathcal{L}_k, d_{\mathcal{E}}) \tag{14.2.2}$$

Remark 14.2.8. The shift by k acts on a curved dg-sheaf $(\mathcal{E}, d_{\mathcal{E}})$ by adding k to the $\mathbb{C}^*_R$-weight of $\mathcal{E}$. If $W = 0$ and the $\mathbb{C}^*_R$-action is trivial, then $[k]$ reduces to the usual homological shift, hence the notation.

Definition 14.2.9. *Let $(\mathcal{E}, d_{\mathcal{E}})$ and $(\mathcal{F}, d_{\mathcal{F}})$ be curved dg-sheaves. Then a* morphism of curved dg-sheaves *is a pair of morphisms of sheaves (f_0, f_1):*

$$\begin{tikzcd}
\mathcal{E}_0 \arrow[r, "f_0"] \arrow[d, bend right, "P_{\mathcal{E}}"'] & \mathcal{F}_0 \arrow[d, bend right, "P_{\mathcal{F}}"'] \\
\mathcal{E}_1 \arrow[r, "f_1"] \arrow[u, bend right, "Q_{\mathcal{E}}"'] & \mathcal{F}_1 \arrow[u, bend right, "Q_{\mathcal{F}}"']
\end{tikzcd} \tag{14.2.3}$$

such that $f_1 \circ P_{\mathcal{E}} = P_{\mathcal{F}} \circ f_0$ and $f_0 \circ Q_{\mathcal{E}} = Q_{\mathcal{F}} \circ f_1$.

Definition 14.2.10. *Let $f, g : (\mathcal{E}, d_{\mathcal{E}}) \longrightarrow (\mathcal{F}, d_{\mathcal{F}})$ be morphisms of curved dg-sheaves. We say f and g are* homotopy equivalent *if there exist morphisms of sheaves $s : \mathcal{E}_0 \longrightarrow \mathcal{F}_1$ and $t : \mathcal{E}_1 \longrightarrow \mathcal{F}_0$ such that:*

$$\begin{aligned} f_0 - g_0 &= Q_{\mathcal{F}} \circ s + t \circ P_{\mathcal{E}} \\ f_1 - g_1 &= P_{\mathcal{F}} \circ t + s \circ Q_{\mathcal{E}}. \end{aligned} \tag{14.2.4}$$

Remark 14.2.11. Definition 14.2.10 is made clearer by the following diagram:

$$\begin{tikzcd}
\mathcal{E}_0 \arrow[r, shift left, "f_0"] \arrow[r, shift right, "g_0"'] \arrow[d, bend right, "P_{\mathcal{E}}"'] \arrow[dr, "s"] & \mathcal{F}_0 \arrow[d, bend right, "P_{\mathcal{F}}"'] \\
\mathcal{E}_1 \arrow[r, shift left, "f_1"] \arrow[r, shift right, "g_1"'] \arrow[u, bend right, "Q_{\mathcal{E}}"'] \arrow[d, bend right, "Q_{\mathcal{E}}"'] \arrow[dr, "t"] & \mathcal{F}_1 \arrow[u, bend right, "Q_{\mathcal{F}}"'] \arrow[d, bend right, "P_Q"'] \\
\mathcal{E}_0 \arrow[r, shift left, "f_0"] \arrow[r, shift right, "g_0"'] \arrow[u, bend right, "P_{\mathcal{E}}"'] & \mathcal{F}_0 \arrow[u, bend right, "P_{\mathcal{F}}"']
\end{tikzcd} \tag{14.2.5}$$

If we view a curved dg-sheaf as a $\mathbb{Z}_2$-graded complex, we recover the usual notion of homotopy equivalence of complexes. The same happens in the case of $W = 0$ and trivial $\mathbb{C}^*_R$-action, where curved dg-sheaves reduce to usual $\mathbb{Z}$-grade complexes.

Definition 14.2.12. *We call* graded D-brane *a curved dg-sheaf* $(\mathcal{E}, d_{\mathcal{E}})$ *such that* $\mathcal{E}$ *is a* $\mathbb{C}^*_R$*-equivariant vector bundle.*

Definition 14.2.13. *Let* $f : (\mathcal{E}, d_{\mathcal{E}}) \longrightarrow (\mathcal{F}, d_{\mathcal{F}})$ *be a morphism of curved dg-sheaves. We call* cone over *f the following curved dg-sheaf:*

$$\mathrm{Cone}(f) := \begin{tikzcd} \mathcal{F}_0 \oplus \mathcal{E}_1 \arrow[d, bend right, "P_f"'] \\ \mathcal{F}_1 \oplus \mathcal{E}_0 \arrow[u, bend right, "Q_f"'] \end{tikzcd} \tag{14.2.6}$$

where P_f and Q_f are defined by:

$$P_f = \begin{pmatrix} P_f & f_1 \\ 0 & -Q_{\mathcal{E}} \end{pmatrix} \quad Q_f = \begin{pmatrix} Q_f & f_0 \\ 0 & -P_{\mathcal{E}} \end{pmatrix} \tag{14.2.7}$$

Remark 14.2.14. In a similar way, one can construct an iterated cone $\mathrm{Cone}(f, g)$ over a pair of morphisms $f : (\mathcal{E}, d_{\mathcal{E}}) \longrightarrow (\mathcal{F}, d_{\mathcal{F}})$, $g : (\mathcal{F}, d_{\mathcal{F}}) \longrightarrow (\mathcal{G}, d_{\mathcal{G}})$ by simply taking

$$\mathrm{Cone}(f, g) := \mathrm{Cone}((g, 0)) \tag{14.2.8}$$

where $(g, 0) : \mathrm{Cone}(f) \longrightarrow \mathcal{G}$.

14.2.1 Categories of curved dg-sheaves

Let us fix a Landau–Ginbzburg model (X, W). To every pair $(\mathcal{E}, d_{\mathcal{E}})$, $(\mathcal{E}, d_{\mathcal{E}})$ of curved dg-sheaves on (X, W) we can associate a curved dg-sheaf $(\mathcal{H}om(\mathcal{E}, \mathcal{F}), d_{\mathcal{H}om(\mathcal{E},\mathcal{F})})$ where $\mathcal{H}om(\mathcal{E}, \mathcal{F})$ is the usual sheaf hom (which is equivariant) and the boundary map is given by

$$d_{\mathcal{H}om(\mathcal{E},\mathcal{F})} := d_{\mathcal{E}^\vee} \otimes \mathrm{Id}_{\mathcal{F}} - \mathrm{Id}_{\mathcal{E}^\vee} \otimes d_{\mathcal{F}} \tag{14.2.9}$$

which has square zero as one can easily check. Therefore, we can construct a dg-category of curved dg-sheaves:

Definition 14.2.15. *Let (X, W) be a Landau–Ginzburg model. Then we call* naïve category of quasi-coherent curved dg-sheaves $QCoh^{nv}_{dg}(X, W)$ *the category described by the following:*

- *Objects: curved dg-sheaves $(\mathcal{E}, d_{\mathcal{E}})$*
- *Morphism: for every pair $(\mathcal{E}, d_{\mathcal{E}})$, $(\mathcal{F}, d_{\mathcal{F}})$ of curved dg-sheaves a curved dg-sheaf*

$$(\mathcal{H}om(\mathcal{E}, \mathcal{F}), d_{\mathcal{H}om(\mathcal{E},\mathcal{F})}). \tag{14.2.10}$$

One can show that these data define indeed a dg-category. In order to construct a derived category out of $QCoh^{nv}_{dg}(X, W)$, we need to define a notion of quasi-isomorphism of curved dg-sheaves. From now on, where this does not impact clarity, let us drop the boundary operator and refer to a curved dg-sheav $(\mathcal{E}, d_{\mathcal{E}})$ simply as $\mathcal{E}$.

Definition 14.2.16. *Let $\mathcal{E}^\bullet = \ldots \longrightarrow \mathcal{E}^{k-1} \xrightarrow{\delta_{k-1}} \mathcal{E}^k \xrightarrow{\delta k} \mathcal{E}^{k+1} \ldots$ be a $[0, N]$-bounded complex of curved dg-sheaves. We say that $\mathcal{E}^\bullet$ is acyclic if it is homotopy equivalent to the iterated cone* $\mathrm{Cone}(\delta_0, \ldots, \delta_N)$.

Definition 14.2.17. *Let (X, W) be a Landau–Ginzburg model. We call* derived category of quasi-coherent curved dg-sheaves *of (X, W) the Verdier quotient*

$$DQCoh_{dg}(X, W) := {[QCoh^{nv}_{dg}(X, W)]} \Big/ {[\mathcal{A}]} \tag{14.2.11}$$

where $\mathcal{A}$ is the full subcategory of acyclic objects in $QCoh^{nv}_{dg}(X, W)$, and we call $DCoh_{dg}(X, W) \subset DQCoh_{dg}(X, W)$ the full subcategory of coherent curved dg-sheaves. Moreover, we call derived category of matrix factorizations *the full subcategory $DMF(X, W) \subset DCoh_{dg}(X, W)$ whose objects are matrix factorizations.*

14.3 Knörrer periodicity

Hereafter we are going to review a result which allows to bridge between the derived category of coherent sheaves of a zero locus of a regular section of a vector bundle and the derived category of matrix factorizations of the associated Landau–Ginzburg model. Such result, called Knörrer periodicity (Shi12, Theorem 3.4), has an immediate application in the context of roofs. This allows us to lift the derived equivalence of a Calabi–Yau pair associated to a roof to an equivalence of matrix factorization categories.

Let us consider a smooth projective variety B and a vector bundle $\mathcal{E}$ over B. Take a regular section $S \in H^0(B, \mathcal{E})$ and call Y its zero locus. Call p the restriction of π to $\pi^{-1}(Y)$ and i the embedding of $\pi^{-1}(Y)$ inside the total space $X = \mathcal{E}^\vee$. Let us collect this information in the following diagram:

$$\begin{tikzcd} \pi^{-1}(Y) \arrow[r, hook, "i"] \arrow[d, "p"] & X \arrow[d, "\pi"] \\ Y \arrow[r, hook] & B \end{tikzcd} \tag{14.3.1}$$

Then, one can construct a Landau–Ginzburg model out of this data in the following way:

- The section $S \in H^0(B, \mathcal{E})$ gives rise to a morphism $\mathcal{O}_B \longrightarrow \mathcal{E}$. Taking the pullback to X of the dual, we get a map $S : \pi^*\mathcal{E}^\vee \longrightarrow \mathcal{O}_X$.
- Consider the section S_B of $\pi^*\mathcal{E}^\vee$ which vanishes exactly on B. As a superpotential, we choose the composition $W := S \circ S_B : X \longrightarrow \mathbb{C}$.
- On X we choose the $\mathbb{C}^*_R$-action which acts with weight two on the coordinate of the fiber, and trivially on the coordinate of the base (on every local trivialization). Hence π is $\mathbb{C}^*_R$-invariant, while S_B has weight zero and S has weight two. Hence, W has weight two as required.

Theorem 14.3.1 (Knörrer periodicity)**.** *Let (X, W) be the Landau–Ginzburg model described above. Then the functor:*

$$i_* p^* : D^b(Y) \longrightarrow \mathrm{DMF}(\mathcal{X}, w)$$

is an equivalence of categories.

14.4 D-branes and roofs: an application of Knörrer periodicity

Let us now consider a roof $X \simeq \mathbb{P}(\mathcal{E}) \simeq \mathbb{P}(\widetilde{\mathcal{E}})$, where $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ are respectively G- and $\widetilde{G}$-homogeneous vector bundles (note that this is different from requiring X to be a homogeneous roof). Then, if we call $\mathcal{X} := \mathcal{E}^\vee$, $\widetilde{\mathcal{X}} := \widetilde{\mathcal{E}}^\vee$, fixing a section $\Sigma \in H^0(X, \mathcal{O}(1,1))$ we can construct two Landau–Ginzburg models $(\mathcal{X}, W)$ and $(\widetilde{\mathcal{X}}, \widetilde{W})$ where the superpotentials are defined as $W = S \circ S_B$ and $\widetilde{W} = \widetilde{S}_{\widetilde{B}} \circ \widetilde{S}$ once we call S and $\widetilde{S}$ the pushforwards of Σ to B and $\widetilde{B}$. Then, if $(Y, \widetilde{Y})$ is a derived equivalent Calabi–Yau pair defined by Σ, we establish the following diagram, where all arrows are equivalences:

$$\begin{tikzcd} \mathrm{DMF}(\mathcal{X}, W) & \mathrm{DMF}(\widetilde{\mathcal{X}}, \widetilde{W}) \\ D^b(Y) \arrow[u] \arrow[r] & D^b(\widetilde{Y}) \arrow[u] \end{tikzcd} \tag{14.4.1}$$

Here the vertical arrows are given by Knörrer periodicity.

For the roof of type A^G_{2k}, for every smooth hyperplane section the two Landau–Ginzburg models as above have been constructed by an explicit GLSM phase transitions described in in terms of variation of GIT with respect to the action of a non abelian group (see Chapter 13.2), and for $k = 2$ the vacuum manifolds are derived equivalent (Proposition 9.5.7). In this context, the fact that the derived equivalence $D^b(Y) \simeq D^b(\widetilde{Y})$ lifts to an equivalence of matrix factorization categories is physically motivated by the fact that D-brane categories of different phases of the same gauged linear sigma model are expected to be equivalent, and such categories of branes are mathematically described with the language of matrix factorizations. It would be an interesting problem to establish a similar picture for other derived equivalent Calabi–Yau pairs arising from roofs.

Bibliography

[Arr96] Arrondo. *Subvarieties of Grassmannians*. 1996. http://www.mat.ucm.es/ arrondo/trento.pdf

[ADM19] Nicholas Addington, Will Donovan, and Ciaran Meachan. *Mukai flops and P-twists*. Journal für die Reine und Angewandte Mathematik. 748, 227-240 (2019).

[ADS15] Nicolas Addington, Will Donovan, Ed Segal. *The Pfaffian–Grassmannian equivalence revisited*. Alg. Geom. 2(3):332–364, 2015. e-Print:ArXiv:1401.3661.

[Bal06] Vikraman Balaji. Lectures on principal bundles. 3rd cycle. Guanajuato (Mexique), 2006, pp.17. cel-00392133

[Bei78] Alexander Beilinson. *Coherent sheaves on* $\mathbb{P}^n$ *and problems in linear algebra* (Russian) Funktsional. Anal. i Prilozhen. 12 (1978), no. 3, 68–69.

[Böh06] Christian Böhning. *Derived categories of coherent sheaves on rational homogeneous manifolds*. Doc. Math. 11 (2006), 261–331 (electronic). MR 2262935 (2008f:14032

[Bon89] Alexiei Bondal. *Representations of associative algebras and coherent sheaves*. Izv. Akad. Nauk SSSR, Ser. Mat., 53 (1989), 25–44; translation in Math. USSR-Izv., 34 (1990), 23–42.

[Bo94] Francis Borceux. *Handbook of Categorical Algebra, Vol. 1: Basic Category Theory*. Cambridge University Press (1994)

[Bor91] Armand Borel. *Linear Algebraic Groups*. Springer-Verlag New York. 1991. DOI: 10.1007/978-1-4612-0941-6

[Bri02] Tom Bridgeland. *Flops and derived categories*. Invent. math. 147, 613–632 (2002). https://doi.org/10.1007/s002220100185

[BC08] Lev Borisov, Andrei Căldăraru. *The Pfaffian-Grassmannian derived equivalence* J. Algebraic Geom. 18 (2009), 201-222 DOI: https://doi.org/10.1090/S1056-3911-08-00496-7

[BCP20] Lev A. Borisov, Andrei Caldararu, Alexander Perry. *Intersections of two Grassmannians in $\mathbb{P}^9$*. Journal für die reine und angewandte Mathematik, Volume 2020, Issue 760, Pages 133–162, eISSN 1435-5345, ISSN 0075-4102, DOI: https://doi.org/10.1515/crelle-2018-0014. e-Print:arXiv:1707.00534.

[BFM19] Marcello Bernardara, Enrico Fatighenti, Laurent Manivel. *Nested varieties of $K3$ type*. e-Print: arXiv:1912.03144

[BFMT17] Vladimiro Bemnedetti, Sara A. Filippini, Laurent Manivel, Fabio Tanturri. *Orbital degeneraci loci and* *applications*. Annali della Scuola normale superiore di Pisa, Classe di scienze. DOI: 10.2422/2036-2145.201804 016.

[BK] Alexey I. Bondal, Mikhail M. Kapranov. *Representable functors, Serre functors, and mutations*. Izv. Akad. Nauk SSSR Ser. Mat., 1989, Volume 53, Issue 6, Pages 1183–1205 (Mi izv1153)

[BM02] Tom Bridgeland, Antony Maciocia. *Fourier-Mukai transforms for K3 and elliptic fibrations*. Journal: J. Algebraic Geom. 11 (2002), 629-657 DOI: https://doi.org/10.1090/S1056-3911-02-00317-X

[BO01] Alexey I. Bondal, Dmitri Orlov. *Reconstruction of a Variety from the Derived Category and Groups of Autoequivalences*. Compositio Mathematica 125, 327–344 (2001). https://doi.org/10.1023/A:1002470302976

[BO02] Alexei Bondal and Dmitri Orlov. *Derived categories of coherent sheaves*. In International Congress of Mathematicians, page 47, 2002.

[BO95] Alexei Bondal, Dmitri Orlov. *Semiorthogonal decomposition for algebraic varieties*. e-Print:arXiv:alg-geom/9506012v1. 1995.

[BV03] Alexey I. Bondal, Michel van den Bergh. (2003). *Generators and representability of functors in commutative and noncommutative geometry*. Moscow Mathematical Journal, 3(1), 1-36.

[Cal05] Andrei Caldararu. *Derived categories of sheaves: a skimming*. e-Print:arXiv:https://arxiv.org/abs/math/0501094

[Cor97] John F. Cornwell. *Group Theory in Physics: An Introduction*. Academic Press, 1997.

[CDGP91] Philip Candelas, Xenia de la Ossa, Paul S. Green, Linda Parkes. *A pair of Calabi–Yau manifolds as an exactly soluble superconformal theory*. Nuclear PhysicsB359 (1991) 21-74 North-Holland.

[CFFGKS08] Ionut Ciocan-Fontanine, David Favero, Jérémy Guéré, Bumsig Kim, Mark Shoemaker. *Fundamental Factorization of a GLSM, Part I: Construction*. e-Print:ArXiv:1802.05247. (2008).

[CGP15] Brian Conrad, Ofer Gabber, Gopal Prasad. (2015). Pseudo-reductive Groups (2nd ed., New Mathematical Monographs). Cambridge: Cambridge University Press. doi:10.1017/CBO9781316092439

[CLNS18] Antoine Chambert-Loir, Johannes Nicaise, Julien Sebag. *Motivic integration*. Progress in Mathematics, Volume 325. DOI: 10.1007/978-1-4939-7887-8

[Don77] Ron Y. Donagi. *On the geometry of Grassmannians*. Duke Math. J. 44 (1977), no. 4, 795–837. doi:10.1215/S0012-7094-77-04436-2. https://projecteuclid.org/euclid.dmj/1077312543

[DK20] Olivier Debarre, Alexander Kuznetsov. *Gushel–*

Mukai varieties: intermediate Jacobians. e-Print:arXiv:2002.05620 (2020)

[DR19] Ivan Dimitrov, Mike Roth. *Intersection multiplicity one for classical groups*. Transformation Groups (2019). https://doi.org/10.1007/s00031-018-9509-2

[Efi18] Alexander I. Efimov. *Some remarks on L-equivalence of algebraic varieties*. Selecta Math. (N.S.), 24:4 (2018), 3753–3762.

[Fon19] Anton Fonarev. *Full exceptional collections on Lagrangian Grassmannians*. e-Print:arXiv:1911.08968

[FI73] Robert M. Fossum, Birger Iversen. *On Picard groups of algebraic fibre spaces*. Journal of Pure and Applied Algebra, Volume 3, Issue 3, September 1973, Pages 269-280

[Ful98] William Fulton. *Intersection theory*. (1998). Springer-Verlag, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, Volume 2

[GM03] Sergei I. Gelfand, Yuri I. Manin. *methods of homological algebra*. Springer Monographs in Mathematics, Ed. 2, Springer-Verlag, Berlin, 2003. https://doi.org/10.1007/978-3-662-12492-5

[GS14] Sergei Galkin, Evgeny Shinder. *The Fano variety of lines and rationality problem for a cubic hypersurface*. e-Print:arXiv:1405.5154

[GS19] Daniel R. Grayson, Michael E. Stillman, Macaulay2 — a software system for research in algebraic geometry (version 1.12), available at: http://www.math.uiuc.edu/Macaulay2/, 2019.

[Hal04] Mihai Halic. *Quotients of affine spaces for actions of reductive groups*. e-Print:arXiv:math/0412278

[Hel62] Sigurdur Helgason. *Differential Geometry and Symmetric Spaces*. 500 pp. Academic Press, New York, 1962. Russian translation, MIR, Moscow, 1964. Published by Amer. Math. Soc. in 2000.

[Huy06] Daniel Huybrechts. *Fourier-Mukai Transforms in Algebraic Geometry*. Oxford Mathematical Monographs, ISSN 0964-9174. Clarendon Press, 2006.

[Huy16] Daniel Huybrechts. *Lectures on K3 Surfaces*. Cambridge Studies in Advanced Mathematics (2016). Cambridge: Cambridge University Press. doi:10.1017/CBO9781316594193

[HL18] Brendan Hassett, Kuan-Wen Lai. *Cremona transformations and derived equivalences of K3 surfaces*. Compositio Math. 154 (2018) 1508-1533. e-Print: arXiv:1612.07751, 2018.

[MS] Kentaro Hori, Sheldon Katz, Albrecht Klemm, Rahul Pandharipande, Richard Thomas, Cumrun Vafa, Ravi Vakil, Eric Zaslow. *Mirror symmetry*. Clay Mathematics Monographs, V. 1. ISBN-10: 0-8218-2955-6

[Har77] Robin Hartshorne. *Algebraic Geometry*. Graduate Texts in Mathematics, Volume 52, ISSN 0072-5285. Springer Science Business Media, 1977.

[HT07] Kentaro Hori, David Tong. *Aspects of Non-Abelian Gauge Dynamics in Two-Dimensional N=(2,2) Theories*. JHEP 0705 (2007) 079.

[IMOU16] Atsushi Ito, Makoto Miura, Shinnosuke Okawa, Kazushi Ueda. *Calabi–Yau complete intersections in G_2-Grassmannians*. e-Print: arXiv:1606.04076

[IMOU19] Atsushi Ito, Makoto Miura, Shinnosuke Okawa, Kazushi Ueda. *The class of the affine line is a zero divisor in the Grothendieck ring: via G_2-Grassmannians*. J. Algebraic Geom. 28 (2019), 245-250 https://www.ams.org/journals/jag/2019-28-02/S1056-3911-2018-00731-3/

[Kan18] Akihiro Kanemitsu. *Mukai pairs and simple K -equivalence*. e-Print: arXiv:1812.05392, 2018

[Kan19] Akihiro Kanemitsu. *Extremal Rays and Nefness of Tangent Bundles*. Michigan Math. J. Volume 68, Issue 2 (2019), 301-322. doi:10.1307/mmj/1549681299

[Kap88] Mikhail Kapranov. *On the derived categories of coherent sheaves on some homogeneous spaces*. Invent. Math. 92 (1988), no. 3, 479–508.

[Kaw02] Yujiro Kawamata. *D-equivalence and K-equivalence*. Journal of Differential Geometry, 61(1):147–171, 2002.

[Kaw08] Yujiro Kawamata. *Derived categories and birational geometry*. Proceedings of Symposia in Pure Mathematics 2009 Volume 80.2 https://doi.org/10.1090/pspum/080.2

[King94] Alastair King. *Moduli of representations of finite dimensional algebras*. Quarterly J. Math. Oxford 45 (1994) 515-530.

[Klei74] Steven Kleiman. *The transversality of a general translate*. Compositio Mathematica, tome 28, no 3 (1974), p. 287-297

[Kol96] János Kollár. *Rational Curves on Algebraic Varieties*. 1996, Springer-Verlag Berlin Heidelberg

[Kuz06b] Alexander Kuznetsov. *Homological projective duality for Grassmannians of lines*. e-Print: arXiv:math/0610957

[Kuz06] Alexander Kuznetsov. *Hyperplane sections and derived categories*. Izv. Ross. Akad. Nauk Ser. Mat. 70 (2006), no. 3, 23–128 (Russian); English translation in Izv. Math. 70 (2006), no. 3, 447–547 MR-2238172

[Kuz07] Alexander Kuznetsov. *Homological projective duality*. Publ.math.IHES 105, 157–220 (2007). https://doi.org/10.1007/s10240-007-0006-8

[Kuz08] Alexander Kuznetsov. *Exceptional collections for Grassmannians of isotropic lines*. Proc. London Math. Soc. (3) 97 (2008) 155–182 Ce2008 London Mathematical Society doi:10.1112/plms/pdm056

[Kuz10] Alexander Kuznetsov. *Derived categories of cubic fourfolds*. “Cohomological and Geometric Approaches to Rationality Problems. New Perspectives” Series: Progress in Mathematics, Vol. 282, Bogomolov, Fedor; Tschinkel, Yuri (Eds.) 2010, X, 314 p. 47 illus., Hardcover ISBN: 978-0-8176-4933-3 A Birkhauser book

[Kuz18] Alexander Kuznetsov. *Derived equivalence of Ito–Miura–Okawa–Ueda Calabi–Yau 3-folds*. J. Math. Soc. Japan, Volume 70, Number 3 (2018), 1007-1013.

[KP19] Alexander Kuznetsov, Alexander Perry. *Categorical joins*. e-Print:arXiv:1804.00144

[Kuz11] Alexander Kuznetsov. *Base change for semiorthogonal decompositions*. Compositio Math. 147 (2011), 852–876. doi:10.1112/S0010437X10005166

[KKM20] Grzegorz Kapustka, Michał Kapustka, Riccardo Moschetti. *Equivalence of K3 surfaces from Verra threefolds*. Kyoto J. Math., advance publication, 24 September 2020. doi:10.1215/21562261-2019-0059. https://projecteuclid.org/euclid.kjm/1600934411

[KR17] Michał Kapustka, Marco Rampazzo. *Torelli problem for Calabi-Yau threefolds with GLSM description*. Communications in Number Theory and Physics 13(4) DOI: 10.4310/CNTP.2019.v13.n4.a2. 2019.

[KR20] Michał Kapustka, Marco Rampazzo. *Mukai duality via roofs of projective bundles*. e-Print:arXiv:2001.06385

[KS18] Alexander Kuznetsov, Evgeny Shinder. *Grothendieck ring of varieties, D- and L-equivalence, and families of quadrics*. Selecta Math, V. 24 (2018), No. 4, pp. 3475–3500. e-Print: arXiv:1612.07193, 2016.

[Laz04a] Robert Lazarsfeld. *Positivity in Algebraic Geometry I*. 2004. A Series of Modern Surveys in Mathematics, vol 48. Springer, Berlin, Heidelberg. http://doi-org-443.webvpn.fjmu.edu.cn/10.1007/978-3-642-18808-4 7

[Laz04b] Robert Lazarsfeld. *Positivity in Algebraic Geometry II*. 2004. A Series of Modern Surveys in Mathematics, vol 49. Springer, Berlin, Heidelberg.

[Li19] Duo Li. *On certain K-equivalent birational maps*. Math. Z. 291, 959–969 (2019). https://doi.org/10.1007/s00209-018-2169-z

[LL03] Michael Larsen, Valery A. Lunts,*Motivic measures and stable birational geometry*. Mosc. Math. J. 3 (2003), no. 1, 85–95, 259.

[LMS75] Alexandru T. Lascu, David Mumford, David Bernard Scott. *The self-intersection formula and the 'formule-clef'*. Math. Proc. Camb. Phil. Soc. (1975), 78, 117 MPCPS 78-10

[LX19] Naichung Conan Leung, Ying Xie. *DK Conjecture for Derived Categories of Grassmannian Flips*. 2019. e-Print:arXiv:1911.07715

[Man17] Laurent Manivel. *Double spinor Calabi-Yau varieties*. Épijournal de Géométrie Algébrique, Volume 3 (April 1, 2019) epiga:5333.

[Mil17] James Milne. *Algebraic Groups: The Theory of Group Schemes of Finite Type over a Field*. Cambridge Studies in Advanced Mathematics. Cambridge University Press. doi:10.1017/9781316711736. 2017

[Mor19] Hayato Morimura. *Derived equivalences for the flops of type C_2 and A_4^G via mutation of semiorthogonal decomposition*. e-Print:arXiv:1812.06413

[Mit01] Stephen A. Mitchell. *Notes on principal bundles and classifying spaces*. Available at http://math.mit.edu/ mbehrens/18.906spring10/prin.pdf

[Muk87] Shigeru Mukai. *Curves, K3 surfaces and Fano 3-folds of genus 10*. Algebraic Geometry and Commutative Algebra in Honor of Masayoshi NAGATA, Kinokuniya, Tokyo, 1987, pp. 357–377.

[Muk88] Shigeru Mukai. *Problems on characterization of the complex projective space*. Birational Geometry of Algebraic Varieties, Open Problems, Katata the 23rd Int'l Symp., Taniguchi Foundation, 1988, pp. 57–60.

[Muk98] Shigeru Mukai. *Duality of polarized K3 surfaces*. Proceedings of Euroconference on Algebraic Geometry, (K. Hulek and M. Reid ed.), Cambridge University Press, 1998, 107122.

[Mum12] David Mumford. *Abelian varieties*. Tata Institute of Fundamental Research Publications Volume: 13; 2012

[MFK94] David Mumford, John Fogarty, Frances Kirwan. *Geometric invariant theory*. Springer-Verlag Berlin Heidelberg. 1994.

[MM86] T. Matsusaka, David Mumford. *Two fundamental theorems on deformations of polarized varieties.* Amer. J. Math., 86, pp. 668–684.

[Nam03] Yoshinori Namikawa. *Mukai flops and derived categories*. J. reine angew. Math. 560 (2003), 65—76

[Nor82] Madhav V. Nori. *The fundamental group-scheme*. Proc. Indian Acad. Sci. (Math. Sci.) 91, 73–122 (1982). https://doi.org/10.1007/BF02967978

[Occ99] Gianluca Occhetta. *Extremal rays of smooth projective varieties*. PhD Thesis, 1999. Available at https://core.ac.uk/download/pdf/150080249.pdf

[Orl92] Dmitri Orlov. *Projective bundles, monoidal transformations, and derived categories of coherent sheaves*. Izv. Akad. Nauk SSSR Ser. Mat., 56 (1992): 852–862; English transl., Russian Acad. Sci. Izv. Math., 41 (1993): 133–141.

[Orl03] Dmitri Orlov. *Triangulated categories of singularities and equivalences between Landau-Ginzburg models*. Sbornik: Mathematics 197.12 (2006): 1827.00–224.

[Ogu01] Keiji Oguiso. *K3 surfaces via almost-primes*. Math. Res. Lett., 9(1):47–63, 2002.

[Orl97] Dmitri O. Orlov. *Equivalences of derived categories and K3 surfaces*. J. Math. Sci. 84 (1997), 1361- 1381.

[Orl03] Dmitri O. Orlov. *Derived categories of coherent sheaves and equivalences between them*. Russian Math. Surveys 58:3 511–591.

[Ott88] Giorgio Ottaviani. *Spinor bundles on quadrics*. Transactions of the American mathematical society, volume 307, number 1, May 1988.

[Ott95] Giorgio Ottaviani. *Rational Homogeneous Varieties*. Course notes in algebraic geometry for SMI, Cortona, Italy. http://web.math.unifi.it/users/ottavian/rathomo/rathomo.pdf, 1995.

[OR17] John C. Ottem, Jørgen V. Rennemo. *A counterexample to the birational Torelli problem for Calabi–Yau threefolds*. J. London Math. Society 2017 DOI:10.1112/jlms.12111. e-Print: arXiv:1706.09952, 2017.

[ORS20] Gianluca Occhetta, Luis E. Solá Conde, Eleonora A. Romano. *Manifolds with two projective bundle structures*. 2020. e-Print:arXiv:2001.06215

[Rød98] Einar A. Rødland. *The Pfaffian Calabi–Yau, its Mirror, and their Link to the Grassmannian*

G(2,7). Compositio Mathematica (2000) 122: 135. https://doi.org/10.1023/A:1001847914402, 2000.

[RS17] Jørgen Vold Rennemo Ed Segal. *Hori-mological projective duality*. Duke Math. J. 168 (2019), no. 11, 2127–2205. doi:10.1215/00127094-2019-0014. https://projecteuclid.org/euclid.dmj/1563328950

[RS00] Kristian Ranestad, Frank-Olaf Schreyer. *Varieties of sums of powers*. J. Reine Angew. Math. 525 (2000) 147–181

[Sam06] Alexander Samokhin. *Some remarks on the derived categories of coherent sheaves on homogeneous spaces*. Journal of the London Mathematical Society 76(1) DOI: 10.1112/jlms/jdm038. 2007.

[Seg15] Ed Segal. *A new 5-fold flop and derived equivalence*. Bulletin of the London Mathematical Society 48(3) DOI: 10.1112/blms/bdw026. 2015.

[Shi12] Ian Shipman. (2012). A geometric approach to Orlov's theorem. Compositio Mathematica, 148(5), 1365-1389. doi:10.1112/S0010437X12000255

[Shi] Evgeny Shinder. *Derived Categories: Lecture 2*. https://e-shinder.staff.shef.ac.uk/Mainz-Lecture-2.pdf

[Toen11] Bertrand Toën. *Lectures on DG–categories*. Topics in Algebraic and Topological K-Theory (2011) 243-301 DOI : 10.1007/978-3-642-15708-0

[Ued19] Kazushi Ueda. *G2−Grassmannians and derived equivalences*. Manuscripta Mathematica. 159(3−4), 549−559 (2019).

[Voi10] Claire Voisin. *Hodge theory and complex algebraic geometry*. Cambridge University Press. 2002. ISBN:9780511615344. https://doi.org/10.1017/CBO9780511615344

[Wey03] Jerzy Weyman. *Cohomology of vector bundles and syzygies*. Cambridge University Press. 2003.

[Wit93] Edward Witten. *Phases of N=2 Theories In Two Dimensions*. Nucl.Phys.B403:159-222,1993